

SCATTERING THEORY FOR THE RADIAL $\dot{H}^{\frac{1}{2}}$ -CRITICAL WAVE EQUATION WITH A CUBIC CONVOLUTION

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ABSTRACT. In this paper, we study the global well-posedness and scattering for the wave equation with a cubic convolution $\partial_t^2 u - \Delta u = \pm(|x|^{-3} * |u|^2)u$ in dimensions $d \geq 4$. We prove that if the radial solution u with life-span I obeys $(u, u_t) \in L_t^\infty(I; \dot{H}_x^{1/2}(\mathbb{R}^d) \times \dot{H}_x^{-1/2}(\mathbb{R}^d))$, then u is global and scatters. By the strategy derived from concentration compactness, we show that the proof of the global well-posedness and scattering is reduced to disprove the existence of two scenarios: soliton-like solution and high to low frequency cascade. Making use of the No-waste Duhamel formula and double Duhamel trick, we deduce that these two scenarios enjoy the additional regularity by the bootstrap argument of [7]. This together with virial analysis implies the energy of such two scenarios is zero and so we get a contradiction.

Key Words: wave-Hartree equation; scattering theory; concentration compactness

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1. INTRODUCTION

This paper is devoted to the study of the Cauchy problem of the energy-subcritical wave equation with a cubic convolution

$$\begin{cases} \partial_t^2 u - \Delta u + f(u) = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^d, \ d \geq 4 \\ (u, u_t)(t_0, x) = (u_0, u_1)(x) \in \dot{H}^{1/2}(\mathbb{R}^d) \times \dot{H}^{-1/2}(\mathbb{R}^d), \end{cases} \quad (1.1)$$

where $f(u) = \mu(V(\cdot) * |u|^2)u$ with $V(x) = |x|^{-3}$, $\mu = \pm 1$ with $\mu = 1$ known as the defocusing case and $\mu = -1$ as the focusing case. Here u is a real-valued function defined in \mathbb{R}^{d+1} , Δ is the Laplacian in \mathbb{R}^d , $V(x)$ is called the potential, and $*$ denotes the spatial convolution in \mathbb{R}^d . Especially for $d = 5$, $V(x) = |x|^{-3}$ is the Newtonian potential.

If the solution u of (1.1) has sufficient decay at infinity and smoothness, it conserves energy:

$$E(u, u_t) = \frac{1}{2} \int_{\mathbb{R}^d} (|\partial_t u|^2 + |\nabla u|^2) dx + \frac{\mu}{4} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(t, x)|^2 |u(t, y)|^2}{|x - y|^3} dx dy = E(u_0, u_1).$$

The equation (1.1) is invariant under the scaling

$$(u, u_t)(t, x) \mapsto (\lambda^{\frac{d-1}{2}} u(\lambda t, \lambda x), \lambda^{\frac{d+1}{2}} u_t(\lambda t, \lambda x)), \quad \forall \lambda > 0. \quad (1.2)$$

One can verify that the only homogeneous L_x^2 -based Sobolev space that is left invariant under (1.2) is $\dot{H}^{\frac{1}{2}}(\mathbb{R}^d) \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)$. And so, it is natural to consider the Cauchy problem with initial data $(u_0, u_1)(x) \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^d) \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)$. We will prove that any radial

maximal-lifespan solution that remains uniformly bounded in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^d) \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)$ must be global and scatter.

First, we introduce some definitions and background materials.

Definition 1.1 (Strong solution). *Let I be a nonempty time interval including t_0 . A function $u : I \times \mathbb{R}^d \rightarrow \mathbb{R}$ is a strong solution to problem (1.1), if*

$$(u, u_t) \in C_t^0(J; \dot{H}^{\frac{1}{2}}(\mathbb{R}^d) \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)), \text{ and } u \in L_{t,x}^{\frac{2(d+1)}{d-1}}(J \times \mathbb{R}^d)$$

for any compact $J \subset I$ and for each $t \in I$ such that

$$\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} = V_0(t - t_0) \begin{pmatrix} u_0(x) \\ u_1(x) \end{pmatrix} - \int_{t_0}^t V_0(t - s) \begin{pmatrix} 0 \\ f(u(s)) \end{pmatrix} ds, \quad (1.3)$$

where

$$V_0(t) = \begin{pmatrix} \dot{K}(t) & K(t) \\ \ddot{K}(t) & \dot{K}(t) \end{pmatrix}, \quad K(t) = \frac{\sin(t\omega)}{\omega}, \quad \omega = (-\Delta)^{1/2}, \quad (1.4)$$

and the dot denotes the time derivative. We refer to the interval I as the lifespan of u . We say that u is a maximal-lifespan solution if the solution cannot be extended to any strictly large interval. We say that u is a global solution if $I = \mathbb{R}$.

The solution lies in the space $L_{t,x}^{\frac{2(d+1)}{d-1}}(I \times \mathbb{R}^d)$ locally in time is natural since by Strichartz estimate in Lemma 2.2 below, the linear flow always lies in this space. We define the scattering size of a solution u to problem (1.1) on a time interval I by

$$\|u\|_{S(I)} := \|u\|_{L_{t,x}^{\frac{2(d+1)}{d-1}}(I \times \mathbb{R}^d)} =: S_I(u). \quad (1.5)$$

Standard arguments show that if a solution u of problem (1.1) is global, with $S_{\mathbb{R}}(u) < +\infty$, then it scatters, i.e. there exist unique $(v_0^\pm, v_1^\pm) \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^d) \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)$ such that

$$\left\| \begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} - V_0(t) \begin{pmatrix} v_0^\pm \\ v_1^\pm \end{pmatrix} \right\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^d) \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)} \longrightarrow 0, \quad \text{as } t \longrightarrow \pm\infty. \quad (1.6)$$

The notion closely associated with scattering is the definition of blowup:

Definition 1.2 (Blowup). *We call that a maximal-lifespan solution $u : I \times \mathbb{R}^d \rightarrow \mathbb{R}$ of problem (1.1) blows up forward in time if there exists a time $\tilde{t}_0 \in I$ such that $\|u\|_{S([\tilde{t}_0, \sup I))} = +\infty$. Similarly, $u(t, x)$ blows up backward in time if there exists a time $\tilde{t}_0 \in I$ such that $\|u\|_{S((\inf I, \tilde{t}_0])} = +\infty$.*

Now we state our main result.

Theorem 1.1. *Assume that $d \geq 4$. Let $u : I \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a radial maximal-lifespan solution to problem (1.1) such that*

$$\|(u, u_t)\|_{L_t^\infty(I; \dot{H}^{\frac{1}{2}}(\mathbb{R}^d) \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d))} < +\infty. \quad (1.7)$$

Then the solution u is global and scatters in the sense of (1.6).

Remark 1.1. *We here focus on the $\dot{H}^{\frac{1}{2}}$ -critical problem, which is corresponding to $V(x) = |x|^{-\gamma}$ with $\gamma = 3$. It is a natural assumption on the dimension $d > \gamma = 3$ for the convolution in the nonlinearity. This is why we are restricted to $d \geq 4$. The radial assumption comes from the technique issue improving the regularity in Section 4 below.*

The impetus to consider this problem stems from a series of recent works for the energy-critical, energy-supercritical and energy-subcritical nonlinear wave equation. To be more precise, let us recall the results for the nonlinear wave equation(NLW)

$$u_{tt} - \Delta u + \mu|u|^p u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d. \quad (1.8)$$

The equation (1.8) is called energy-critical if $p = \frac{4}{d-2}$, energy-supercritical if $p > \frac{4}{d-2}$ and energy-subcritical if $p < \frac{4}{d-2}$. The energy-critical equations have been the most widely studied instances of NLW, since the rescaling

$$(u, u_t)(t, x) \mapsto (\lambda^{\frac{2}{p}} u(\lambda t, \lambda x), \lambda^{\frac{2}{p}+1} u_t(\lambda t, \lambda x)), \quad p = \frac{4}{d-2} \quad (1.9)$$

leaves invariant the energy of solutions, which is defined by

$$E(u, u_t) = \frac{1}{2} \int_{\mathbb{R}^d} (|\partial_t u|^2 + |\nabla u|^2) dx + \frac{\mu}{p+2} \int_{\mathbb{R}^d} |u|^{p+2} dx,$$

and is a conserved quantity for equation(1.8). For the defocusing energy-critical NLW (1.8), Grillakis [13] proved that the Cauchy problem of equation (1.8) with the $\dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ initial data is global well posedness, we refer the readers to [1], [14], [38] and [43] for the scattering theory and the high dimensional case. In particular, Tao derived a exponential type spacetime bound in [43]. In the above papers, their methods rely heavily on the classical finite speed of propagation (i.e. the monotonic local energy estimate on the light cone)

$$\int_{|x| \leq R-t} e(t, x) dx \leq \int_{|x| \leq R} e(0, x) dx, \quad t > 0 \quad (1.10)$$

where

$$e(t, x) := \frac{1}{2} |u_t|^2 + \frac{1}{2} |\nabla u|^2 + \frac{d-2}{2d} |u|^{\frac{2d}{d-2}}, \quad (1.11)$$

and Morawetz estimate

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} \frac{|u|^{\frac{2d}{d-2}}}{|x|} dx dt \leq C(E(u_0, u_1)). \quad (1.12)$$

However, the Morawetz estimate fails for the focusing energy-critical NLW. Kenig and Merle [17] first employed sophisticated “concentrated compactness + rigidity method” to obtain the dichotomy-type result under the assumption that $E(u_0, u_1) < E(W, 0)$, where W denotes the ground state of the elliptic equation

$$\Delta W + |W|^{\frac{4}{d-2}} W = 0.$$

The analogs for the focusing energy-critical nonlinear Schrödinger equation in the radial case for dimensions 3 and 4 have also been established by Kenig and Merle [16]. Thereafter, Bulut et.al [4] extended the above result in [17] to higher dimensions. While we refer the readers to Killip and Visan [25] for the focusing energy-critical Schrödinger

equation in high dimensions. This was proven by making use of minimal counterexamples derived from the concentration-compactness approach to induction on energy.

For the \dot{H}^{s_c} -critical NLW (1.8) with $p \neq \frac{4}{d-2}$, both the Morawetz estimate and energy conservation fails. It is hard to prove global well posedness and scattering of equation (1.8) in the space $\dot{H}^{s_c} \times \dot{H}^{s_c-1}$. Up to now, we do not know how to treat the large-data case since there does not exist any a priori control of a critical norm. The first result in this direction is due to Kenig-Merle [18], where they studied the $\dot{H}^{\frac{1}{2}}$ -critical Schrödinger equation in \mathbb{R}^3 . For the defocusing energy-supercritical NLW in odd dimensions, Kenig and Merle [19, 20] proved that if the radial solution u is a priori bounded in the critical Sobolev space, that is

$$(u, u_t) \in L_t^\infty(I; \dot{H}_x^{s_c}(\mathbb{R}^d) \times \dot{H}_x^{s_c-1}(\mathbb{R}^d)), \quad s_c := \frac{d}{2} - \frac{2}{p} > 1,$$

then u is global and scatters. Later, Killip and Visan [25] showed the result in \mathbb{R}^3 for the non-radial solutions by making use of Huygens principal and so called “localized double Duhamel trick”. We refer to [3, 26, 31] for some high dimensional cases. Recently, Duyckaerts, Kenig and Merle [11] obtain such result for the focusing energy-supercritical NLW with radial solution in three dimension. Their proof relies on the compactness/rigidity method, pointwise estimates on compact solutions obtained in [19], and the channels of energy method developed in [9]. Furthermore, by exploiting the double Duhamel trick, Dodson and Lawrie [8] extend the result in [11] to dimension five. We also refer reader to [24, 29, 34] for the defocusing energy-supercritical Schrödinger equation.

The methods employed in the energy-supercritical NLW also lead to the study of the energy-subcritical NLW. In fact, using the channels of energy method pioneered in [10, 11], Shen [39] proved the analog result of [19] for $2 < p < 4$ with $d = 3$ in both defocusing and focusing case. However, the channels of energy method does not work so effectively for $p \leq 2$. Recently, by virial based rigidity argument and improving addition regularity for the minimal counterexamples, Dodson and Lawrie [7] extended the result of [39] to $\sqrt{2} < p \leq 2$. The problem of having monotonicity formulae at a different regularity to the critical conservation laws is a difficulty intrinsic to the nonlinear wave equation. In order to enable to utilize the monotonicity formulae, one need improve the regularity for the almost periodic solutions. In [7, 39], they use the double Duhamel trick to show that almost periodic solutions belong to energy space $\dot{H}_x^1(\mathbb{R}^3) \times L_x^2(\mathbb{R}^3)$. The main difficult is that the decay rate of the linear solution is not enough to guarantee the double Duhamel formulae converges. However, the weighted decay available from radial Sobolev embedding can supply the additional decay to guarantee the double Duhamel formulae converges. Thus, one need the radial assumption in [7, 39]. This is different from cubic Schrödinger equation [18], where no radial assumption is made. This is due to the Lin-Strauss Morawetz inequality

$$\int_I \int_{\mathbb{R}^3} \frac{|u(t, x)|^4}{|x|} dx dt \lesssim \sup_{t \in I} \|u(t, x)\|_{\dot{H}_x^{\frac{1}{2}}(\mathbb{R}^3)}^2,$$

which is scaling critical with the cubic Schrödinger equation.

For the Cauchy problem of the nonlocal NLW (1.1) with $V(x) = |x|^{-\gamma}$, making use of the argument developed by Strauss [41, 42] and Pecher [37], Mochizuki [33] showed that if $d \geq 3$, $2 \leq \gamma < \min\{d, 4\}$, then the global well-posedness and scattering results

with small initial data hold in the energy space $H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. For the large initial data, it is difficult since there are the absence of the classical finite speed of propagation for (1.1) and the positive properties $G(u) > 0$ with

$$G(u) = f(u)\bar{u} - 2 \int_0^{|u|} f(r)dr, \quad f(u) = (V * |u|^2)u, \quad V(x) = |x|^{-\gamma}.$$

It plays an important role in establishing the classical Morawetz-type estimates in [35]. Hence, one can not utilize the classical methods in [13, 14, 38] to prove the GWP scattering for the defocusing energy-critical wave equation (1.1) with $\gamma = 4$. While, inspired by the strategy derived from concentration compactness [16, 17] and the new-type Morawetz-type estimate in [36], the authors in [32] showed GWP and scattering result simultaneously for the defocusing energy-critical wave-Hartree equation (1.1) with $V(x) = |x|^{-4}$.

In this paper, we continue the investigations carried out in [32] concerning the long-time behavior of the solution of wave-Hartree, but for $\dot{H}^{1/2}$ -critical wave-Hartree (that is, $\gamma = 3$) in both defocusing and focusing case. Compared with [32], the $\dot{H}^{1/2}$ -critical problem is much more difficult due to the failure of energy conservation law and the Morawetz estimate. By using compactness approach, modifying the argument of improving addition regularity as in [7] and employing the symmetries of the non-local nonlinearity $(|x|^{-3} * |u|^2)u$, we prove that if the radial solution u with life-span I satisfies $(u, u_t) \in L_t^\infty(I; \dot{H}^{\frac{1}{2}}(\mathbb{R}^d) \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d))$, then u is global and scatters.

Now, let us turn to an outline of the arguments establishing Theorem 1.1.

• **The outline of the proof of Theorem 1.1.** Before we can address the global-in-time theory for the problem (1.1), we need to have a good local-in-time theory in place. In particular, we have the following

Theorem 1.2 (Local well-posedness). *Let $d \geq 4$. Assume that $(u_0, u_1) \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^d) \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)$ and $t_0 \in \mathbb{R}$, then there exists a unique maximal-lifespan solution $u : I \times \mathbb{R}^d \rightarrow \mathbb{R}$ to problem (1.1) with initial data (u_0, u_1) . This solution also has the following properties:*

- (1) (Local existence) I is an open neighborhood of t_0 .
- (2) (Blow up criterion) If $\sup(I)$ is finite, then u blows up forward in time in the sense of Definition 1.2. If $\inf(I)$ is finite, then u blows up backward in time.
- (3) (Scattering) If $\sup(I) = +\infty$ and u does not blow up forward in time, then u scatters forward in time in the sense (1.6). Conversely, given $(v_0^+, v_1^+) \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^d) \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)$ there is a unique solution to problem (1.1) in a neighborhood of infinity such that (1.6) holds.
- (4) There exists a $\delta = \delta(d, \|(u_0, u_1)\|_{\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}})$ such that if

$$\|\dot{K}(t - t_0)u_0 + K(t - t_0)u_1\|_{S(I)} < \delta,$$

then, there exists a unique solution $u : I \times \mathbb{R}^d \rightarrow \mathbb{R}$ to the equation in (1.1) with initial data $(u(t_0), \dot{u}(t_0))$ such that $(u, \dot{u}) \in C(I; \dot{H}^{\frac{1}{2}}(\mathbb{R}^d) \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d))$, and

$$\|u\|_{S(I)} \leq 2\delta, \quad \|u\|_{S^{\frac{1}{2}}(I)} < +\infty,$$

where $\|u\|_{S(I)}$ is defined in (1.5) and

$$\|u\|_{S^{\frac{1}{2}}(I)} := \sup_{(q,r) \in \Lambda_0} \{ \|\nabla|^s u\|_{L_t^q(I; L_x^r)} : \frac{1}{q} + \frac{d}{r} = \frac{d}{2} - \left(\frac{1}{2} - s\right), 0 \leq s \leq \frac{1}{2} \},$$

with Λ_0 being defined in Definition 2.1 below.

- (5) (Small data implying scattering) If $\|(u_0, u_1)\|_{\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}}$ is sufficiently small, then u is a global solution which does not blow up either forward or backward in time, and satisfies that

$$\|u\|_{S(\mathbb{R})} \lesssim \|(u_0, u_1)\|_{\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}}.$$

Theorem 1.2 follows from Strichartz estimate in Lemma 2.2 below and the standard fixed argument in [5]. Closely related to the continuous dependence on the data, an essential tool for concentration compactness arguments is the following stability theory.

Lemma 1.1 (Stability). *Let I be a time interval, and let \tilde{u} be a function on $I \times \mathbb{R}^d$ which is a near-solution to problem (1.1) in the sense that*

$$\tilde{u}_{tt} - \Delta \tilde{u} = -f(\tilde{u}) + e \quad (1.13)$$

for some function e . Assume that

$$\begin{aligned} \|\tilde{u}\|_{S(I)} &\leq M, \\ \|(\tilde{u}, \partial_t \tilde{u})\|_{L_t^\infty(I; \dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}})} &\leq E \end{aligned} \quad (1.14)$$

for some constant $M, E > 0$, where $S(I)$ is defined in (1.5). Let $t_0 \in I$, and let $(u(t_0), u_t(t_0)) \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^d) \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)$ be close to $(\tilde{u}(t_0), \tilde{u}_t(t_0))$ in the sense that

$$\|(u(t_0) - \tilde{u}(t_0), u_t(t_0) - \tilde{u}_t(t_0))\|_{\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}} \leq \epsilon \quad (1.15)$$

and assume also that the error term obeys

$$\|e\|_{L_{t,x}^{\frac{2(d+1)}{d+3}}(I \times \mathbb{R}^d)} \leq \epsilon \quad (1.16)$$

for some small $0 < \epsilon < \epsilon_1 = \epsilon_1(M, E)$. Then, we conclude that there exists a solution $u : I \times \mathbb{R}^d \rightarrow \mathbb{R}$ to problem (1.1) with initial data $(u(t_0), u_t(t_0))$ such that

$$\begin{aligned} \|u - \tilde{u}\|_{L_t^q(I; L_x^r)} &\leq C(M, E), \\ \|u\|_{L_t^q(I; L_x^r)} &\leq C(M, E), \\ \|u - \tilde{u}\|_{S(I)} &\leq C(M, E)\epsilon^c, \end{aligned} \quad (1.17)$$

where constant $c = c(d, M, E) > 0$ and (q, r) is admissible pair and satisfies $\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - \frac{1}{2}$.

Lemma 1.1 follows from well-known arguments, which are in fact similar in spirit to the arguments used to prove local well-posedness. For an example of such an argument in the context of wave equation with a cubic convolution, see for example [32].

With the local theory in hand, we are now in a position to sketch the proof of Theorem 1.1 by the compactness procedure as in [17, 27]. For any $0 \leq E_0 < +\infty$, we

define

$$L(E_0) := \sup \left\{ S_I(u) : u : I \times \mathbb{R}^d \rightarrow \mathbb{R} \text{ such that } \sup_{t \in I} \|(u, u_t)\|_{\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}}^2 \leq E_0 \right\},$$

where the supremum is taken over all solutions $u : I \times \mathbb{R}^d \rightarrow \mathbb{R}$ to problem (1.1) satisfying $\sup_{t \in I} \|(u, u_t)\|_{\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}}^2 \leq E_0$. Thus, $L : [0, +\infty) \rightarrow [0, +\infty)$ is a non-decreasing function. Furthermore, we obtain from “Small data implying scattering”

$$L(E_0) \lesssim E_0^{\frac{1}{2}} \quad \text{for } E_0 \leq \eta_0^2.$$

From Lemma 1.1, we see that L is continuous. Therefore, there must exist a unique critical element $E_c \in (0, +\infty]$ such that $L(E_0) < +\infty$ for $E_0 < E_c$ and $L(E_0) = +\infty$ for $E_0 \geq E_c$. In particular, if $u : I \times \mathbb{R}^d \rightarrow \mathbb{R}$ is a maximal-lifespan solution to problem (1.1) such that $\sup_{t \in I} \|(u, u_t)\|_{\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}} < E_c$, then u is global and moreover,

$$S_{\mathbb{R}}(u) \leq L(\|(u, u_t)\|_{L_t^\infty(\mathbb{R}; \dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}})}).$$

Therefore, the proof of Theorem 1.1 is equivalent to show $E_c = +\infty$. We argue by contradiction. The failure of Theorem 1.1 would imply the existence of very special class of solutions. Our goal is to exclude such special class of solutions. Before making further reductions, we give the definition of the almost periodic solution.

Definition 1.3 (Almost periodic solution). *Let $d \geq 4$, a solution $(u, u_t)(t)$ to problem (1.1) with lifespan I is called almost periodic modulo symmetries if $(u, u_t)(t)$ is bounded in $\dot{H}_x^{1/2}(\mathbb{R}^d) \times \dot{H}_x^{-1/2}(\mathbb{R}^d)$ and there exist functions $N(t) : I \rightarrow \mathbb{R}^+$, $x(t) : I \rightarrow \mathbb{R}^d$ and $C(\eta) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that for all $t \in I$ and $\eta > 0$,*

$$\int_{|x-x(t)| \geq \frac{C(\eta)}{N(t)}} \left(|\nabla|^{1/2} u(t, x)|^2 + |\nabla|^{-1/2} u_t(t, x)|^2 \right) dx \leq \eta \quad (1.18)$$

and

$$\int_{|\xi| \geq C(\eta)N(t)} \left(|\xi| \cdot |\hat{u}(t, \xi)|^2 + |\xi|^{-1} \cdot |\hat{u}_t(t, \xi)|^2 \right) d\xi \leq \eta. \quad (1.19)$$

We refer to the function $N(t)$ as the frequency scale function of the solution u , to $x(t)$ as the spatial center function, and to $C(\eta)$ as the compactness modules function.

We remark that for radial almost periodic solutions, one can take the spatial center function $x(t) \equiv 0$, see [27].

By using the Bahouri-Gérard type profile decomposition from [1, 2] and the above stability result, we deduce that the failure of Theorem 1.1 would imply the existence of the radial almost periodic solutions.

Theorem 1.3 (Reduction to radial almost periodic solutions, [16, 24, 30]). *Assume $E_c < +\infty$. Then there exists a radial maximal-lifespan solution $u : I \times \mathbb{R}^d \rightarrow \mathbb{R}$ to problem (1.1) such that*

- (1) u is radial almost periodic modulo symmetries;
- (2) u blows up both forward and backward in time;

- (3) u has the minimal $L_t^\infty \dot{H}^{\frac{1}{2}}$ -norm among all blowup solutions. More precisely, let $v : J \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a maximal-lifespan solution which blows up in at least one time direction, then

$$\sup_{t \in I} \|(u, u_t)(t)\|_{\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}} \leq \sup_{t \in J} \|(v, v_t)(t)\|_{\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}}.$$

The reduction to almost periodic solutions is now widely regarded as a standard technique in the study of dispersive equations at critical regularity. Their existence was first proved in the pioneering work by Karaani [21] for the mass-critical NLS. Kenig and Merle [16] adapted the argument to the energy-critical NLS, and first applied this to study the wellposedness and scattering problem. Since then, the technique has proven to be extremely useful, see [25–27, 30] for many more examples of these techniques. In particular, for a very nice introduction to concentration compactness methods, one should refer to [27, 45].

With Theorem 1.3 in place, we can now make some refinements to the class of solutions that we consider. A rescaling argument and possibly time reversal as in [23, 44] show that we can restrict our attention to radial almost periodic solutions that do not escape to arbitrarily high frequencies on at least half of their maximal lifespan $[0, \infty)$.

Theorem 1.4 (Two enemies, [27, 30, 44]). *Suppose Theorem 1.1 fails. Then there exists a radial maximal-lifespan solution $u : (T_-, T_+) \times \mathbb{R}^d \rightarrow \mathbb{R}$, which is radial almost periodic modulo symmetries with $S_{(T_-, T_+)}(u) = +\infty$. Moreover, we can also ensure that $T_+(u) = +\infty$, $T_- < 0$ and the frequency scale function $N(t)$ matches one of the following two scenarios:*

- (1) (Soliton-like solution) We have $N(t) = 1$ for all $t \in \mathbb{R}$.
- (2) (High-to-low frequency cascade) We have

$$\sup_{t \in \mathbb{R}^+} N(t) \leq 1, \text{ and } \lim_{t \rightarrow \infty} N(t) = 0.$$

In view of this theorem, our goal is to preclude the possibilities of all the scenarios.

A further manifestation of the minimality of u as a blow-up solution is the absence of the scattered wave at the endpoints of the lifespan I . Formally, we have the following Duhamel formula, which plays an important role in proving the additional regularity. One can refer to [6, 27, 30] for the proof.

Lemma 1.2 (No waste Duhamel formula). *Let $u : I \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a maximal-lifespan solution to the equation in (1.1) which is almost periodic modulo symmetries. Then, for all $t \in I$, there holds that*

$$\begin{aligned} \begin{pmatrix} u(t) \\ u_t(t) \end{pmatrix} &= \lim_{T \nearrow \sup(I)} \int_t^T V_0(t-s) \begin{pmatrix} 0 \\ f(u)(s) \end{pmatrix} ds \\ &= - \lim_{T \searrow \inf(I)} \int_T^t V_0(t-s) \begin{pmatrix} 0 \\ f(u)(s) \end{pmatrix} ds, \end{aligned} \tag{1.20}$$

as weak limits in $\dot{H}_x^{1/2}(\mathbb{R}^d) \times \dot{H}_x^{-1/2}(\mathbb{R}^d)$. Here $V_0(t)$ is defined as in (1.4).

In view of this lemma and note that the minimal $L_t^\infty(\dot{H}_x^{\frac{1}{2}} \times \dot{H}_x^{-\frac{1}{2}})$ -norm blowup solution is localized in both physical and frequency space, we can show that it admits additional regularity by the bootstrap argument and double Duhamel trick.

Theorem 1.5 (Additional Regularity). *Let $u : (T_-, +\infty) \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a radial solution to equation in (1.1) which is almost periodic modulo symmetries in the sense of Theorem 1.4. And assume that $N(t) \leq 1$ on $t \in \mathbb{R}^+$, then for each $t \in \mathbb{R}^+$, there holds*

$$\|(u, u_t)\|_{\dot{H}_x^1(\mathbb{R}^d) \times L_x^2(\mathbb{R}^d)} \lesssim \begin{cases} N(t)^{\frac{1}{2}} & \text{if } d \geq 5, \\ N(t)^{\frac{1}{3}} & \text{if } d = 4. \end{cases} \quad (1.21)$$

It follows from this theorem that the high-to-low frequency cascade solution satisfies

$$\lim_{t \rightarrow \infty} \|(u, u_t)\|_{\dot{H}_x^1(\mathbb{R}^d) \times L_x^2(\mathbb{R}^d)} \lesssim \lim_{t \rightarrow +\infty} N(t)^{\frac{1}{3}} = 0,$$

which implies that

$$E(u, u_t)(t) \rightarrow 0, \text{ as } t \rightarrow +\infty.$$

This together with Theorem 2.1 below precludes the existence of the high-to-low frequency cascade solution.

Moreover, for the soliton-like solution, we have

Proposition 1.1. *Let $u : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a radial solution to equation in (1.1) which is almost periodic modulo symmetries in the sense of Theorem 1.4 with $N(t) \equiv 1$. Then the set $K := \{(u, u_t)(t) : t \in \mathbb{R}\}$ is precompact in $(\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}) \cap (\dot{H}^1 \times L^2)$.*

We will utilize this proposition and the virial identity to prove that the energy of such solution is exact zero. Thus, we exclude the existence of the soliton-like solution. We refer to Section 3 for more details.

The paper is organized as follows. In Section 2, as preliminaries, we gather notations and recall the Strichartz estimate for wave equation and some useful lemmas. In Section 3, we exclude two scenarios in the sense of Theorem 1.4 under the assumption that Theorem 1.5 and Proposition 1.1 hold. In Section 4, we show Theorem 1.5 and Proposition 1.1, and so we conclude Theorem 1.1.

2. PRELIMINARIES

2.1. Notations and useful lemmas. First, we give some notations which will be used throughout this paper. To simplify our inequalities, we introduce the symbols \lesssim, \sim, \ll . If X, Y are nonnegative quantities, we write either $X \lesssim Y$ or $X = O(Y)$ to denote the estimate $X \leq CY$ for some C , and $X \sim Y$ to denote the estimate $X \lesssim Y \lesssim X$. We use $X \ll Y$ to mean $X \leq cY$ for some small constant c . We use $C \gg 1$ to denote various large finite constants, and $0 < c \ll 1$ to denote various small constants. For every $r, 1 \leq r \leq \infty$, we denote by $\|\cdot\|_r$ the norm in the Lebesgue space $L^r = L^r(\mathbb{R}^d)$ and by r' the conjugate exponent defined by $\frac{1}{r} + \frac{1}{r'} = 1$.

The Fourier transform on \mathbb{R}^d is defined by

$$\hat{f}(\xi) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx,$$

giving rise to the fractional differentiation operator $|\nabla|^s$ defined by

$$|\nabla|^s f(x) := \mathcal{F}_\xi^{-1}(|\xi|^s \hat{f}(\xi))(x).$$

This helps us to define the homogeneous and inhomogeneous Sobolev norms

$$\|f\|_{\dot{W}_x^{s,p}(\mathbb{R}^d)} := \| |\nabla|^s f \|_{L_x^p(\mathbb{R}^d)}.$$

We will also need the Littlewood-Paley projection operators. Specifically, let $\varphi(\xi)$ be a smooth bump function adapted to the ball $|\xi| \leq 2$ which equals 1 on the ball $|\xi| \leq 1$. For $k \in \mathbb{Z}$, we define the Littlewood-Paley operators

$$\begin{aligned} \widehat{P_{\leq k} f}(\xi) &:= \varphi\left(\frac{\xi}{2^k}\right) \widehat{f}(\xi), \\ \widehat{P_{> k} f}(\xi) &:= \left(1 - \varphi\left(\frac{\xi}{2^k}\right)\right) \widehat{f}(\xi), \\ \widehat{P_k f}(\xi) &:= \left(\varphi\left(\frac{\xi}{2^k}\right) - \varphi\left(\frac{2\xi}{2^k}\right)\right) \widehat{f}(\xi). \end{aligned}$$

Similarly we can define $P_{< k}$, $P_{\geq k}$, and $P_{m < \cdot \leq k} = P_{\leq k} - P_{\leq m}$.

The Littlewood-Paley operators commute with derivative operators, the free propagator, and the conjugation operation. They are self-adjoint and bounded on every L_x^p and \dot{H}_x^s space for $1 \leq p \leq \infty$ and $s \geq 0$, moreover, they also obey the following Bernstein estimates

$$\begin{aligned} \| |\nabla|^{\pm s} P_k f \|_{L^p} &\sim 2^{\pm ks} \|P_k f\|_{L^p}, \\ \|P_{\leq k} f\|_{L^q} &\lesssim 2^{(\frac{d}{p} - \frac{d}{q})k} \|P_{\leq k} f\|_{L^p}, \\ \|P_k f\|_{L^q} &\lesssim 2^{(\frac{d}{p} - \frac{d}{q})k} \|P_k f\|_{L^p}, \end{aligned}$$

where $1 \leq p \leq q \leq \infty$.

Next, we record here a refinement of the Sobolev embedding for radial functions, which will be of use in Section 4.

Lemma 2.1 (Radial Sobolev embedding, [44]). *Let $d \geq 1$, $1 \leq q \leq \infty$, $0 < s < d$, and $\beta \in \mathbb{R}$ obey the conditions*

$$\beta > -\frac{d}{q}, \quad 1 \leq \frac{1}{p} + \frac{1}{q'} \leq 1 + s$$

and the scaling condition

$$\beta + s = \frac{d}{p} - \frac{d}{q}$$

with at most one of the equalities

$$p = 1, \quad p = \infty, \quad q = 1, \quad q = \infty, \quad \frac{1}{p} + \frac{1}{q'} = 1 + s$$

holding. Then for any spherically symmetric function $f \in \dot{W}^{s,p}(\mathbb{R}^d)$, we have

$$\|r^\beta f\|_{L^q(\mathbb{R}^d)} \lesssim \| |\nabla|^s f \|_{L^p(\mathbb{R}^d)}. \quad (2.1)$$

The Strichartz estimates involve the following definitions:

Definition 2.1 (Admissible pairs). *A pair of Lebesgue space exponents (q, r) are called wave admissible in \mathbb{R}^{1+d} , denoted by $(q, r) \in \Lambda_0$ when $q, r \geq 2$, and*

$$\frac{2}{q} \leq (d-1)\left(\frac{1}{2} - \frac{1}{r}\right), \quad \text{and} \quad (q, r, d) \neq (2, \infty, 3). \quad (2.2)$$

Now we recall the following Strichartz estimates.

Lemma 2.2 (Strichartz estimates, [12, 15, 28, 40]). *Let I be a compact time interval and let $u : I \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a solution to the forced wave equation*

$$u_{tt} - \Delta u + F = 0$$

with initial data $(u, \dot{u})|_{t=t_0} = (u_0, u_1) \in \dot{H}^s \times \dot{H}^{s-1}$ for $s > 0$. Then

$$\|u\|_{L_t^q(I, L_x^r)} \lesssim \|(u(t_0), \partial_t u(t_0))\|_{\dot{H}^s \times \dot{H}^{s-1}} + \|F\|_{L_t^{q'_1}(I, L_x^{r'_1})},$$

where (q, r) and (q_1, r_1) are admissible pairs satisfying

$$\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - s = \frac{1}{q'_1} + \frac{d}{r'_1} - 2.$$

Furthermore, we have the frequency localized Strichartz estimate

$$\|(\nabla_x, \partial_t)P_k u\|_{L_t^{q_1}(I, L_x^{r_1})} \lesssim 2^{k\mu_1} \|P_k(u_0, u_1)\|_{\dot{H}^1 \times L^2} + 2^{k\mu_2} \|P_k F\|_{L_t^{q'_2}(I; L_x^{r'_2})}, \quad (2.3)$$

where (q_1, r_1) and (q_2, r_2) are admissible pairs and satisfy

$$\mu_1 = \delta(r_1) - \frac{1}{q_1}, \quad \mu_2 = \mu_1 + \delta(r_2) - \frac{1}{q_2}, \quad \delta(r) := d\left(\frac{1}{2} - \frac{1}{r}\right).$$

2.2. Blow-up for non-positive energies. We recall that in the case of the focusing equation, any nontrivial solution with non-positive energy must blow-up in both time directions. Such result was first proved in Killip, Stovall and Visan [22] for the solutions to NLW (1.8). By the same argument as deriving Theorem 3 in [22] with the extended causality (c.f. Lemma 2.5, [32]) which plays role of the finite speed propagation, we also have

Proposition 2.1 (Blow-up for non-positive energies). *Let $u(t, x)$ be a solution to problem (1.1) with $\mu = -1$ and with maximal interval of existence $I_{\max} = (T_-, T_+)$. If $E(u, u_t) \leq 0$, then $(u, u_t)(t)$ is either identically zero or blows up in finite time in both time directions, i.e. $T_- > -\infty$ and $T_+ < +\infty$.*

3. EXTINCTION OF TWO SCENARIOS

In this section, we preclude two scenarios in the sense of Theorem 1.4 under the assumption that Theorem 1.5 and Proposition 1.1 holds. And we will prove Theorem 1.5 and Proposition 1.1 in the next section.

3.1. High to low frequency cascade. First, we preclude the high to low frequency cascade solution by making use of Theorem 1.5.

Theorem 3.1 (No high to low frequency cascade). *Let $d \geq 4$. Then there are no radial almost periodic solutions $u : (T_-, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}$ to problem (1.1) with $N(t) \leq 1$ on \mathbb{R}^+ such that*

$$\|u\|_{S((T_-, \infty))} = +\infty \quad (3.1)$$

and

$$\lim_{t \rightarrow \infty} N(t) = 0. \quad (3.2)$$

Proof. We argue by contradiction. Assume that u were such a solution. Using Theorem 1.5, we obtain

$$\|(u, u_t)\|_{\dot{H}_x^1 \times L_x^2} \lesssim N(t)^{\frac{1}{3}}, \quad \forall t \in \mathbb{R}^+. \quad (3.3)$$

Combining this inequality with (3.2), we get

$$\lim_{t \rightarrow \infty} \|(u, u_t)\|_{\dot{H}_x^1 \times L_x^2} \lesssim \lim_{t \rightarrow \infty} N(t)^{\frac{1}{3}} = 0.$$

On the other hand, by the Hölder inequality, the Hardy-Littlewood-Sobolev inequality, the Sobolev embedding and interpolation, we estimate the potential energy as follows

$$\|(|x|^{-3} * |u|^2)|u|^2\|_{L_x^1} \lesssim \|u\|_{L_x^{\frac{4d}{2d-3}}}^4 \lesssim \|u\|_{\dot{H}^{\frac{3}{4}}}^4 \lesssim \|u\|_{\dot{H}^{1/2}}^2 \|u\|_{\dot{H}^1}^2 \rightarrow 0, \quad \text{as } t \rightarrow \infty.$$

Hence, we have

$$\lim_{t \rightarrow \infty} E(u, u_t)(t) = 0.$$

This together with energy conservation yields

$$E(u_0, u_1) = 0.$$

Combining this with Proposition 2.1 if $\mu = -1$, we get $u(t) \equiv 0$. This contradicts with (3.1). And so we complete the proof of Theorem 3.1. \square

3.2. The soliton-like solution. Next, we turn to exclude the soliton-like solution under the assumption that Proposition 1.1 holds.

Theorem 3.2 (No soliton-like solution). *Let $d \geq 4$. Then, there are no radial almost periodic solutions $u : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ to problem (1.1) with $N(t) \equiv 1$ on \mathbb{R} such that*

$$\|u\|_{S(\mathbb{R})} = \infty. \quad (3.4)$$

First, we need to establish the following virial identity.

Lemma 3.1 (virial identity). *Let $\vec{u} := (u, u_t) \in (\dot{H}^1 \times L^2) \cap (\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}})$ be a solution to problem (1.1). Then, for any $R > 0$, we have*

$$\begin{aligned} & \frac{d}{dt} \left\langle u_t, \chi_R \left(\frac{d-1}{2} u + r u_r \right) \right\rangle \\ &= - \frac{1}{|\mathbb{S}^{d-1}|} E(\vec{u}) + \int (1 - \chi_R) \left[\frac{1}{2} |u_r|^2 + \frac{1}{2} |u_t|^2 \pm \frac{d-1}{2} (|x|^{-3} * |u|^2) |u|^2 \right] r^{d-1} dr \\ & \quad - \int \left[\frac{1}{2} |u_r|^2 + \frac{1}{2} |u_t|^2 \right] \chi'_R r^d dr - \frac{d-1}{2} \int u u_r \chi'_R r^{d-1} dr \\ & \quad \pm \int (|x|^{-3} * |u|^2) u u_r (1 - \chi_R) r^d dr, \end{aligned} \quad (3.5)$$

where

$$\langle f, g \rangle \triangleq \int_0^\infty f(r) g(r) r^{d-1} dr,$$

and $\chi_R(r) = \chi(r/R)$, $\chi \in C_c^\infty(\mathbb{R}^+)$ with

$$\chi(r) = \begin{cases} 1, & 0 \leq r \leq 1, \\ 0, & r \geq 2. \end{cases}$$

Proof. We compute by Leibniz rule

$$\begin{aligned} & \frac{d}{dt} \left\langle u_t, \chi_R \left(\frac{d-1}{2} u + r u_r \right) \right\rangle \\ &= \int u_{tt} \chi_R \left(\frac{d-1}{2} u + r u_r \right) r^{d-1} dr + \int u_t \chi_R \left(\frac{d-1}{2} u_t + r \partial_r u_t \right) r^{d-1} dr \\ &\triangleq I_1 + I_2. \end{aligned} \tag{3.6}$$

The computation of I_2 : Using integration by part, we obtain

$$\begin{aligned} I_2 &= \frac{d-1}{2} \int |u_t|^2 \chi_R r^{d-1} dr + \int u_t \partial_r u_t \chi_R r^d dr \\ &= \frac{d-1}{2} \int |u_t|^2 \chi_R r^{d-1} dr - \frac{1}{2} \int |u_t|^2 \partial_r (\chi_R r^d) dr \\ &= -\frac{1}{2} \int |u_t|^2 \chi_R r^{d-1} dr - \frac{1}{2} \int |u_t|^2 \chi'_R r^d dr. \end{aligned}$$

The computation of I_1 : By equation (1.1) and the radial assumption, we rewrite

$$u_{tt} = \Delta u \mp (|x|^{-3} * |u|^2)u = \partial_r^2 u + \frac{d-1}{r} \partial_r u \mp (|x|^{-3} * |u|^2)u.$$

Plugging this equality into I_1 , one has

$$\begin{aligned} I_1 &= \int \partial_r^2 u \chi_R \left(\frac{d-1}{2} u + r u_r \right) r^{d-1} dr + (d-1) \int \partial_r u \chi_R \left(\frac{d-1}{2} u + r u_r \right) r^{d-2} dr \\ &\quad \mp \int (|x|^{-3} * |u|^2) u \chi_R \left(\frac{d-1}{2} u + r u_r \right) r^{d-1} dr \\ &\triangleq I_{11} + I_{12} \pm I_{13}. \end{aligned}$$

First, we consider the contribution of $I_{11} + I_{12}$. Integrating by part, we derive that

$$\begin{aligned} I_{11} &= \frac{d-1}{2} \int \partial_r^2 u u \chi_R r^{d-1} dr + \int \partial_r^2 u u_r \chi_R r^d dr \\ &= -\frac{d-1}{2} \int |u_r|^2 \chi_R r^{d-1} dr - \frac{d-1}{2} \int u_r u \partial_r (\chi_R r^{d-1}) dr - \frac{1}{2} \int |u_r|^2 \partial_r (\chi_R r^d) dr \\ &= -\frac{2d-1}{2} \int |u_r|^2 \chi_R r^{d-1} dr + \frac{d-1}{4} \int |u|^2 \partial_r^2 (\chi_R r^{d-1}) dr - \frac{1}{2} \int |u_r|^2 \chi'_R r^d dr, \end{aligned}$$

and

$$\begin{aligned} I_{12} &= \frac{(d-1)^2}{2} \int \partial_r u u \chi_R r^{d-2} dr + (d-1) \int |u_r|^2 \chi_R r^{d-1} dr \\ &= -\frac{(d-1)^2}{4} \int |u|^2 (\chi_R r^{d-2})_r dr + (d-1) \int |u_r|^2 \chi_R r^{d-1} dr. \end{aligned}$$

Hence,

$$\begin{aligned} I_{11} + I_{12} &= -\frac{1}{2} \int |u_r|^2 \chi_R r^{d-1} dr + \frac{d-1}{4} \int |u|^2 \partial_r (\chi'_R r^{d-1}) dr - \frac{1}{2} \int |u_r|^2 \chi'_R r^d dr \\ &= -\frac{1}{2} \int |u_r|^2 \chi_R r^{d-1} dr - \frac{d-1}{2} \int u u_r \chi'_R r^{d-1} dr - \frac{1}{2} \int |u_r|^2 \chi'_R r^d dr. \end{aligned}$$

Second, we turn to consider the contribution of the term I_{13} . Since

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{x \cdot (x - y)}{|x - y|^5} |u(x)|^2 |u(y)|^2 dx dy = - \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{y \cdot (x - y)}{|x - y|^5} |u(x)|^2 |u(y)|^2 dx dy,$$

we obtain

$$\int_{\mathbb{R}^d} (|x|^{-3} * |u|^2) u(x) x \cdot \nabla u(x) dx = \left(-\frac{d}{2} + \frac{3}{4} \right) \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(x)|^2 |u(y)|^2}{|x - y|^3} dx dy.$$

This symmetrical identity implies

$$\begin{aligned} I_{13} &= \frac{d-1}{2} \int \chi_R (|x|^{-3} * |u|^2) |u|^2 r^{d-1} dr + \int (|x|^{-3} * |u|^2) u u_r r^d dr \\ &\quad - \int (|x|^{-3} * |u|^2) u u_r (1 - \chi_R) r^d dr \\ &= \frac{1}{4} \int (|x|^{-3} * |u|^2) |u|^2 r^{d-1} dr - \frac{d-1}{2} \int (1 - \chi_R) (|x|^{-3} * |u|^2) |u|^2 r^{d-1} dr \\ &\quad - \int (|x|^{-3} * |u|^2) u u_r (1 - \chi_R) r^d dr. \end{aligned}$$

Plugging the above computations into (3.6), we get

$$\begin{aligned} &\frac{d}{dt} \langle u_t, \chi_R (u + r u_r) \rangle \\ &= - \int \left[\frac{1}{2} |u_r|^2 + \frac{1}{2} |u_t|^2 \pm \frac{1}{4} (|x|^{-3} * |u|^2) |u|^2 \right] r^{d-1} dr \\ &\quad + \int \left[\frac{1}{2} |u_r|^2 + \frac{1}{2} |u_t|^2 \pm \frac{d-1}{2} (|x|^{-3} * |u|^2) |u|^2 \right] (1 - \chi_R) r^{d-1} dr \\ &\quad - \int \left[\frac{1}{2} |u_r|^2 + \frac{1}{2} |u_t|^2 \right] \chi'_R r^d dr - \frac{d-1}{2} \int u u_r \chi'_R r^{d-1} dr \\ &\quad \pm \int (|x|^{-3} * |u|^2) u u_r (1 - \chi_R) r^d dr \\ &= - \frac{1}{|\mathbb{S}^{d-1}|} E(\vec{u}) + \int (1 - \chi_R) \left[\frac{1}{2} |u_r|^2 + \frac{1}{2} |u_t|^2 \pm \frac{d-1}{2} (|x|^{-3} * |u|^2) |u|^2 \right] r^{d-1} dr \\ &\quad - \int \left[\frac{1}{2} |u_r|^2 + \frac{1}{2} |u_t|^2 \right] \chi'_R r^d dr - \frac{d-1}{2} \int u u_r \chi'_R r^{d-1} dr \\ &\quad \pm \int (|x|^{-3} * |u|^2) u u_r (1 - \chi_R) r^d dr. \end{aligned}$$

Thus, we complete the proof of this lemma. \square

The proof of Theorem 3.2. We argue by contradiction. Assume that u were such a solution. We claim that

$$\forall \eta > 0, \quad E(u, u_t) \leq C \eta^{\frac{1}{2}}. \quad (3.7)$$

Combining this claim with Proposition 2.1 if $\mu = -1$, we get $u(t) \equiv 0$. This contradicts with (3.4). Thus, we preclude the soliton solution in the sense of Theorem 1.4 under the assumption that Proposition 1.1 holds. \square

Next, we use Proposition 1.1 and Lemma 3.1 to show the claim (3.7).

• **The proof of claim (3.7):** By Proposition 1.1, we deduce that for any $\eta > 0$, there exists a $R_0 = R_0(\eta) > 0$ such that for any $R \geq R_0(\eta)$

$$\int_R^{+\infty} [u_t^2 + u_r^2](t) r^{d-1} dr < \eta. \quad (3.8)$$

This inequality together with Sobolev embedding

$$\dot{H}^{1/2}(\mathbb{R}^d) \cap \dot{H}^1(\mathbb{R}^d) \hookrightarrow L_x^{\frac{4d}{2d-3}}(\mathbb{R}^d)$$

yields

$$\int_R^{+\infty} |u(t)|^{\frac{4d}{2d-3}} r^{d-1} dr < \eta. \quad (3.9)$$

Hence, using the Hölder inequality and the Hardy-Littlewood-Sobolev inequality, we obtain

$$\begin{cases} \left| \int_0^{+\infty} (1 - \chi_R) \left(\frac{1}{2} u_t^2 + \frac{1}{2} u_r^2 \pm \frac{d-1}{2} (|x|^{-3} * |u|^2) |u|^2 \right) r^{d-1} dr \right| \leq C\eta \\ \left| \int_0^{+\infty} \left(\frac{1}{2} u_t^2 + \frac{1}{2} u_r^2 \right) r \chi'_R r^{d-1} dr \right| \lesssim \eta. \end{cases} \quad (3.10)$$

On the other hand, by Hölder's inequality, Hardy's inequality, the radial Sobolev inequality and Proposition 1.1, we get

$$\begin{aligned} \left| \int_0^{+\infty} u u_r r \chi'_R r^{d-2} dr \right| &\lesssim \left(\int_R^{2R} u_r^2 r^{d-1} dr \right)^{\frac{1}{2}} \left(\int_R^{2R} u^2 r^{d-3} dr \right)^{\frac{1}{2}} \\ &\lesssim \eta^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} \frac{u^2}{|x|^2} dx \right)^{\frac{1}{2}} \\ &\lesssim \eta^{\frac{1}{2}} \|u\|_{\dot{H}_x^1(\mathbb{R}^d)} \lesssim \eta^{\frac{1}{2}}, \end{aligned}$$

and

$$\begin{aligned} \left| \int_0^{+\infty} (|x|^{-3} * |u|^2) u u_r (1 - \chi_R) r^d dr \right| &\lesssim \| |x|^{-3} * |u|^2 \|_{L_x^{\frac{d}{2}}(\mathbb{R}^d)} \left(\int_R^{+\infty} u_r^2 r^{d-1} dr \right)^{\frac{1}{2}} \| |x| u \|_{L_x^{\frac{2d}{d-4}}(\mathbb{R}^d)} \\ &\lesssim \|u\|_{\dot{H}_x^{\frac{1}{2}}(\mathbb{R}^d)}^2 \eta^{\frac{1}{2}} \|\nabla u\|_{L_x^2(\mathbb{R}^d)} \\ &\lesssim \eta^{\frac{1}{2}}. \end{aligned}$$

Integrating the inequality (3.5) from 0 to T in time and setting $R = T \gg R_0(\eta)$, one has

$$\begin{aligned} E(u, u_t) &\leq C\eta^{\frac{1}{2}} + \frac{1}{T} \left| \langle u_t(t), \chi_R \left(\frac{d-1}{2} u + r u_r \right)(t) \rangle \right|_0^T \\ &\leq C\eta^{\frac{1}{2}} + \frac{C}{T} \int_0^T |u_t(T)| \cdot |u(T)| r^{d-1} dr + \frac{C}{T} \int_0^T |u_t(0)| \cdot |u(0)| r^{d-1} dr \\ &\quad + \frac{C}{T} \int_0^T |u_t(T)| \cdot |u_r(T)| r^d dr + \frac{C}{T} \int_0^T |u_t(0)| \cdot |u_r(0)| r^d dr. \end{aligned}$$

On the other hand, by the Hölder inequality, the Sobolev embedding and (3.8), we get

$$\begin{aligned} \frac{1}{T} \int_0^T |u_t| \cdot |u| r^{d-1} dr &\leq \frac{1}{T} \left(\int_0^T |u_t|^2 r^{d-1} dr \right)^{\frac{1}{2}} \left(\int_0^T |u|^{\frac{2d}{d-1}} r^{d-1} dr \right)^{\frac{d-1}{2d}} \left(\int_0^T r^{d-1} dr \right)^{\frac{1}{2d}} \\ &\leq \frac{C}{T^{\frac{1}{2}}} \|u_t\|_{L_x^2(\mathbb{R}^d)} \|u\|_{\dot{H}_x^{\frac{1}{2}}(\mathbb{R}^d)}, \end{aligned}$$

and

$$\begin{aligned} \frac{1}{T} \int_0^T |u_t| \cdot |u_r| r^d dr &\leq \frac{R_0(\eta)}{T} \int_0^{R_0(\eta)} |u_t| \cdot |u_r| r^{d-1} dr + \frac{1}{T} \int_{R_0(\eta)}^T |u_t| \cdot |u_r| r^d dr \\ &\leq \frac{R_0(\eta)}{T} \|u_t\|_2 \|u\|_{\dot{H}^1} + \left(\int_{R_0(\eta)}^{+\infty} u_t^2 r^{d-1} dr \right)^{\frac{1}{2}} \left(\int_{R_0(\eta)}^{+\infty} u_r^2 r^{d-1} dr \right)^{\frac{1}{2}} \\ &\leq \frac{R_0(\eta)}{T} + \eta. \end{aligned}$$

Thus,

$$E(u_0, u_1) \leq C\eta^{\frac{1}{2}} + \frac{C}{T^{\frac{1}{2}}} + C\frac{R_0(\eta)}{T}.$$

Letting $T \rightarrow +\infty$, we obtain the claim (3.7).

In sum, it suffices to prove Theorem 1.5 and Proposition 1.1 which will be shown in the next section.

4. ADDITIONAL REGULARITY

As stated in Section 3, it remains to show Theorem 1.5 and Proposition 1.1. More precisely, we need to prove that the solutions which are radial almost periodic modulo symmetries enjoy the additional regularity.

4.1. Proof of Theorem 1.5. In this subsection, we will divide two steps to prove Theorem 1.5. First, we show that the almost periodic solutions in Theorem 1.4 lie in $\dot{H}^{\frac{2}{3}}(\mathbb{R}^d) \times \dot{H}^{-\frac{1}{3}}(\mathbb{R}^d)$ when $t \in \mathbb{R}^+$ by bootstrap argument. And then we utilize this result to show Theorem 1.5.

Theorem 4.1. *Let $u : (T_-, +\infty) \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a radial solution of problem (1.1) which is almost periodic modulo symmetries in the sense of Theorem 1.4. Then for any $t_0 \in \mathbb{R}^+$, we have*

$$\|(u, u_t)(t_0)\|_{\dot{H}_x^{\frac{2}{3}} \times \dot{H}_x^{-\frac{1}{3}}} \lesssim N(t_0)^{\frac{1}{6}}. \quad (4.1)$$

Before showing this theorem, we need the refined Strichartz-type estimate in a small time interval.

Lemma 4.1 (Refined Strichartz-type estimate). *Let u be as in Theorem 4.1 and $J_\delta(t_0) := [t_0 - \frac{\delta}{N(t_0)}, t_0 + \frac{\delta}{N(t_0)}]$. Then, for any $\eta > 0$, there exists a $\delta = \delta(\eta) > 0$ such that for all $t_0 \in \mathbb{R}^+$*

$$\|u\|_{S(J_\delta(t_0))} \lesssim \eta. \quad (4.2)$$

Proof. Without loss of generality, we assume that $t_0 = 0$. By Duhamel formula, we get

$$\|u\|_{S(J_\delta(0))} \lesssim \|L(t)(u_0, u_1)\|_{S(J_\delta(0))} + \left\| \int_0^t L(t-s)(0, \pm(|x|^{-3} * |u|^2)u) ds \right\|_{S(J_\delta(0))}, \quad (4.3)$$

where $L(t)(f, g) := \dot{K}(t)f + K(t)g$ and $K(t)$ is defined by (1.4). For the free part, using Strichartz estimate and (1.18), we have

$$\|L(t)P_{\geq \log_2(C(\eta)N(0))}(u_0, u_1)\|_{S(J_\delta(0))} \lesssim \|P_{\geq \log_2(C(\eta)N(0))}(u_0, u_1)\|_{\dot{H}^{1/2} \times \dot{H}^{-1/2}} \leq \eta. \quad (4.4)$$

On the other hand, we obtain by the Bernstein inequality,

$$\|L(t)P_{\leq \log_2(C(\eta)N(0))}(u_0, u_1)\|_{L_x^q} \lesssim (C(\eta)N(0))^{\frac{1}{q}} \|(u_0, u_1)\|_{\dot{H}^{1/2} \times \dot{H}^{-1/2}}, \quad q = \frac{2(d+1)}{d-1}.$$

Integrating this inequality in time, we obtain

$$\|L(t)P_{\leq \log_2(C(\eta)N(0))}(u_0, u_1)\|_{S(J_\delta(0))} \lesssim C(\eta)^{\frac{1}{q}} \delta^{\frac{1}{q}}. \quad (4.5)$$

For the inhomogenous part, by the Strichartz estimate, the Hölder inequality and the Hardy-Littlewood-Sobolev inequality, we have

$$\begin{aligned} & \left\| \int_0^t L(t-s)(0, \pm(|x|^{-3} * |u|^2)u) ds \right\|_{S(J_\delta(0))} \\ & \lesssim \|(|x|^{-3} * |u|^2)u\|_{L_t^{\frac{q}{2}} L_x^{r_1'}(J_\delta(0) \times \mathbb{R}^d)} \quad \left(q = \frac{2(d+1)}{d-1}, r_1 = \frac{2d(d+1)}{d^2-5} \right) \\ & \lesssim \|u\|_{L_t^\infty \dot{H}^{\frac{1}{2}}} \|u\|_{S(J_\delta(0))}^2. \end{aligned}$$

Plugging the above estimates into (4.3), we deduce that

$$\|u\|_{S(J_\delta(0))} \lesssim \eta + C(\eta)^{\frac{1}{q}} \delta^{\frac{1}{q}} + \|u\|_{S(J_\delta(0))}^2.$$

Hence, we obtain by the bootstrap argument

$$\|u\|_{S(J_\delta(0))} \lesssim \eta.$$

This ends the proof of Lemma 4.1. \square

Now we use the above lemma to show Theorem 4.1.

Proof of Theorem 4.1. By translation invariance in time, we may assume $t_0 = 0$. Then, we aim to prove

$$\|(u, u_t)(0)\|_{\dot{H}^{\frac{2}{3}} \times \dot{H}^{-\frac{1}{3}}} \lesssim N(0)^{\frac{1}{6}}. \quad (4.6)$$

Since

$$\|(u, u_t)(0)\|_{\dot{H}^{\frac{2}{3}} \times \dot{H}^{-\frac{1}{3}}}^2 \simeq \sum_{k \in \mathbb{Z}} \|P_k(u, u_t)(0)\|_{\dot{H}^{\frac{2}{3}} \times \dot{H}^{-\frac{1}{3}}}^2,$$

we only need to construct a frequency envelope $\{\alpha_k(0)\}_{k \in \mathbb{Z}}$ satisfying

$$\begin{cases} \|P_k(u, u_t)(0)\|_{\dot{H}^{\frac{2}{3}} \times \dot{H}^{-\frac{1}{3}}} \lesssim 2^{\frac{1}{6}k} \alpha_k(0), \\ \left\| \{2^{\frac{1}{6}k} \alpha_k(0)\}_{k \in \mathbb{Z}} \right\|_{\ell^2} \lesssim N(0)^{\frac{1}{6}}. \end{cases} \quad (4.7)$$

Proposition 4.1. *Let $\eta > 0$ be a small constant and let $J_\delta(0)$ be as in Lemma 4.1. Define*

$$\begin{cases} a_k := 2^{\frac{k}{2}} \|P_k u\|_{L_t^\infty(J_\delta(0), L_x^2)} + 2^{-\frac{k}{2}} \|P_k u_t\|_{L_t^\infty(J_\delta(0), L_x^2)} + 2^{\frac{k}{4}} \|P_k u\|_{L_t^{q_1}(J_\delta(0), L_x^{r_1})} \\ a_k(0) := 2^{\frac{k}{2}} \|P_k u(0)\|_{L_x^2} + 2^{-\frac{k}{2}} \|P_k u_t(0)\|_{L_x^2}, \end{cases} \quad (4.8)$$

where $\frac{1}{q_1} + \frac{d}{r_1} = \frac{d}{2} - \frac{1}{4}$, $q_1 = \frac{2d+1}{d-1}$, and

$$\alpha_k := \sum_j 2^{-\frac{1}{5}|j-k|} a_j, \quad \alpha_k(0) := \sum_j 2^{-\frac{1}{5}|j-k|} a_j(0). \quad (4.9)$$

It follows from the definition that

$$\left\| P_k(u, u_t)(0) \right\|_{\dot{H}^{\frac{2}{3}} \times \dot{H}^{-\frac{1}{3}}} \simeq 2^{\frac{k}{6}} \left\| P_k(u, u_t)(0) \right\|_{\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}} \simeq 2^{\frac{k}{6}} a_k(0) \lesssim 2^{\frac{k}{6}} \alpha_k(0). \quad (4.10)$$

Then, we have

$$a_k \lesssim a_k(0) + \eta^2 \sum_{j \geq k-3} 2^{\frac{k-j}{4}} a_j \quad (4.11)$$

$$\alpha_k \lesssim \alpha_k(0). \quad (4.12)$$

Proof. We first use the frequency localized Strichartz estimate (2.3) to get

$$a_k \lesssim 2^{\frac{k}{2}} \|P_k u(0)\|_{L_x^2} + 2^{-\frac{k}{2}} \|P_k u_t(0)\|_{L_x^2} + 2^{\frac{k}{4}} \left\| P_k[(|x|^{-3} * |u|^2) u] \right\|_{L_t^{q_2'} L_x^{r_2'}(J_\delta(0) \times \mathbb{R}^d)},$$

where $q_2 = \frac{(2d+1)(d+1)}{4d+3}$, $\delta(r_2) - \frac{1}{q_2} = \frac{3}{4}$. Observing that

$$P_k[(|x|^{-3} * |P_{\leq k-4} u|^2) P_{\leq k-4} u] = 0,$$

and

$$\left\| (|x|^{-3} * |u|^2) P_j u \right\|_{L_t^{q_2'} L_x^{r_2'}} + \left\| (|x|^{-3} * (u \cdot P_j u)) u \right\|_{L_t^{q_2'} L_x^{r_2'}} \lesssim \|P_j u\|_{L_t^{q_1} L_x^{r_1}} \|u\|_{S(J_\delta(0))}^2 \lesssim \eta^2 2^{-\frac{j}{4}} a_j,$$

we obtain by (4.2)

$$a_k \lesssim a_k(0) + \eta^2 \sum_{j \geq k-3} 2^{\frac{k-j}{4}} a_j,$$

as desired estimate in (4.11).

By (4.11), we have

$$\alpha_k = \sum_j 2^{-\frac{|j-k|}{5}} a_j \lesssim \sum_j a_j(0) 2^{-\frac{|j-k|}{5}} + \eta^2 \sum_j 2^{-\frac{|j-k|}{5}} \sum_{j_1 \geq j-3} 2^{\frac{j-j_1}{4}} a_{j_1}.$$

This inequality together with

$$\begin{cases} \sum_{j_1 \leq k} \sum_{j \leq j_1+3} 2^{\frac{j-j_1}{4}} 2^{\frac{j-k}{5}} a_{j_1} \lesssim \sum_{j_1 \leq k} 2^{\frac{j_1-k}{5}} a_{j_1} \lesssim \alpha_k \\ \sum_{j_1 > k} \sum_{j \leq j_1+3} 2^{\frac{j-j_1}{4}} 2^{-\frac{|j-k|}{5}} a_{j_1} \lesssim \sum_{j_1 > k} [2^{-\frac{j_1-k}{4}} + 2^{-\frac{j_1-k}{5}}] a_{j_1} \lesssim \alpha_k \end{cases} \quad (4.13)$$

yields

$$\alpha_k \lesssim \alpha_k(0) + \eta^2 \alpha_k. \quad (4.14)$$

On the other hand, by constructions, we know that $\alpha_k(0)$ is uniformly bounded in k . And so (4.12) follows by (4.14). We conclude Proposition 4.1. \square

Remark 4.1. *From the proof of Proposition 4.1, we also obtain*

$$2^{\frac{k}{2}} \left\| P_k \int_0^{\frac{\delta}{N(0)}} \frac{e^{-it\sqrt{-\Delta}}}{\sqrt{-\Delta}} (|x|^{-3} * |u|^2) u(t) dt \right\|_{L_x^2} \lesssim \eta^2 \sum_{j \geq k-3} 2^{\frac{k-j}{4}} a_j. \quad (4.15)$$

Now we turn to estimate the term in large time interval. Define

$$\chi(x) \in C_c^\infty(\mathbb{R}^d) \quad \chi(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| \geq 2. \end{cases} \quad (4.16)$$

Lemma 4.2 (Large time). *For any $s_0 \in (0, 1/2)$, we have*

$$\int_{\delta/N(0)}^\infty \left\| \frac{e^{-it\sqrt{-\Delta}}}{\sqrt{-\Delta}} (1 - \chi) \left(\frac{8x}{|t|} \right) (|x|^{-3} * |u|^2) u \right\|_{\dot{H}^{\frac{1}{2}+s_0}} dt \lesssim N(0)^{s_0} \delta^{-s_0}. \quad (4.17)$$

In particularly, there exists a sequence $\{b_k\}_{k \in \mathbb{Z}}$ such that

$$\left\| P_k \left(\int_{\delta/N(0)}^\infty \frac{e^{-i\tau\sqrt{-\Delta}}}{\sqrt{-\Delta}} (1 - \chi) \left(\frac{8x}{|\tau|} \right) (|x|^{-3} * |u|^2) u d\tau \right) \right\|_{\dot{H}^{\frac{1}{2}}} \lesssim 2^{-\frac{k}{6}} b_k, \quad (4.18)$$

and

$$\|\{b_k\}_{k \in \mathbb{Z}}\|_{\ell^2} \lesssim N(0)^{\frac{1}{6}}. \quad (4.19)$$

Proof. First, we use the Sobolev embedding to get

$$\begin{aligned} & \left\| \frac{e^{-it\sqrt{-\Delta}}}{\sqrt{-\Delta}} (1 - \chi) \left(\frac{8x}{|t|} \right) (|x|^{-3} * |u|^2) u \right\|_{\dot{H}^{\frac{1}{2}+s_0}} \\ &= \left\| (1 - \chi) \left(\frac{8x}{|t|} \right) (|x|^{-3} * |u|^2) u \right\|_{\dot{H}^{-\frac{1}{2}+s_0}} \\ &\lesssim \left\| (1 - \chi) \left(\frac{8x}{|t|} \right) (|x|^{-3} * |u|^2) u \right\|_{L^p} \quad \left(\frac{d}{p} = \frac{d}{2} + \frac{1}{2} - s_0 \right). \end{aligned} \quad (4.20)$$

Next, by the Hölder inequality and the radial Sobolev embedding, we obtain

$$\begin{aligned} \left\| (1 - \chi) \left(\frac{8x}{|t|} \right) (|x|^{-3} * |u|^2) u \right\|_{L^p} &\lesssim \left\| (1 - \chi) \left(\frac{8x}{|t|} \right) u \right\|_{L^q} \| |x|^{-3} * |u|^2 \|_{L^{\frac{d}{2}}} \\ &\lesssim |t|^{-1-s_0} \| |x|^{1+s_0} u \|_{L^q} \| u \|_{L^{\frac{2d}{d-1}}}^2 \\ &\lesssim |t|^{-1-s_0} \| u \|_{\dot{H}^{\frac{1}{2}}}^3, \end{aligned}$$

where $q = \frac{2d}{d-3-2s_0}$, $\frac{1}{p} = \frac{1}{q} + \frac{2}{d}$. Plugging this inequality into (4.20) yields (4.17).

The inequality (4.18) follows by taking

$$s_0(k) = \begin{cases} \frac{1}{4}, & 2^k \geq N(0) \\ \frac{1}{8}, & 2^k < N(0) \end{cases} \quad b_k = 2^{-k(s_0(k) - \frac{1}{6})} N(0)^{s_0(k)},$$

in (4.17). □

Since the first inequality in (4.7) was already established in (4.10), it remains to prove the second inequality in (4.7). By the definition of $\alpha_k(0)$, we only need to estimate $a_k(0)$. For this purpose, we denote

$$v = u + \frac{i}{\sqrt{-\Delta}} u_t, \quad (4.21)$$

then

$$\|v\|_{\dot{H}^1} \simeq \|(u, u_t)(t)\|_{\dot{H}^1 \times L^2}, \text{ and } a_k(0) = \|P_k v(0)\|_{\dot{H}^{\frac{1}{2}}}.$$

Since $u_{tt} - \Delta u = \pm(|x|^{-3} * |u|^2)u$, v satisfies

$$v_t = u_t + \frac{i}{\sqrt{-\Delta}} u_{tt} = u_t + \frac{i}{\sqrt{-\Delta}} [\Delta u \pm (|x|^{-3} * |u|^2)u] = -i\sqrt{-\Delta}v \pm \frac{i}{\sqrt{-\Delta}} (|x|^{-3} * |u|^2)u.$$

Hence, for $T \in (T_-, 0)$, we have

$$v(0) = e^{iT\sqrt{-\Delta}} v(T) \pm \frac{i}{\sqrt{-\Delta}} \int_T^0 e^{i\tau\sqrt{-\Delta}} (|x|^{-3} * |u|^2)u(\tau) d\tau.$$

Fixing $T_1 \in (T_-, 0)$, and using both the Duhamel formula and no-waste Duhamel formula, we have

$$\begin{aligned} a_k(0)^2 &= \|P_k v(0)\|_{\dot{H}^{\frac{1}{2}}}^2 = \langle P_k v(0), P_k v(0) \rangle_{\dot{H}^{\frac{1}{2}}} \\ &= \left\langle P_k(e^{iT_1\sqrt{-\Delta}} v(T_1)), P_k v(0) \right\rangle_{\dot{H}^{\frac{1}{2}}} \\ &\quad + \lim_{T_2 \rightarrow \infty} \left\langle \frac{i}{\sqrt{-\Delta}} P_k \left(\int_{T_1}^0 e^{it\sqrt{-\Delta}} (|x|^{-3} * |u|^2)u(t) dt \right), \frac{\pm i}{\sqrt{-\Delta}} P_k \int_0^{T_2} e^{i\tau\sqrt{-\Delta}} (|x|^{-3} * |u|^2)u(\tau) d\tau \right\rangle_{\dot{H}^{\frac{1}{2}}} \\ &\triangleq I_1 + \lim_{T_2 \rightarrow \infty} I_2, \end{aligned} \quad (4.22)$$

where $\langle f, g \rangle_{\dot{H}^{\frac{1}{2}}} = \int |\nabla|^{\frac{1}{2}} f |\nabla|^{\frac{1}{2}} \bar{g} dx$.

• **The estimate of I_1 :** We have by the no-waste Duhamel's formula

$$\lim_{T_1 \rightarrow T_-} I_1 = \lim_{T_1 \rightarrow T_-} \left\langle P_k(e^{iT_1\sqrt{-\Delta}} v(T_1)), P_k v(0) \right\rangle_{\dot{H}^{\frac{1}{2}}} = 0. \quad (4.23)$$

• **The estimate of I_2 :** Define

$$A := P_k \left(\int_{-\delta/N(0)}^0 \frac{ie^{it\sqrt{-\Delta}}}{\sqrt{-\Delta}} (|x|^{-3} * |u|^2)u(t) dt + \int_{T_1}^{-\delta/N(0)} \frac{ie^{it\sqrt{-\Delta}}}{\sqrt{-\Delta}} (1-\chi)\left(\frac{8x}{|t|}\right) (|x|^{-3} * |u|^2)u dt \right)$$

and

$$B := P_k \left(\int_{T_1}^{-\delta/N(0)} \frac{ie^{it\sqrt{-\Delta}}}{\sqrt{-\Delta}} \chi\left(\frac{8x}{|t|}\right) (|x|^{-3} * |u|^2)u dt \right).$$

Similarly,

$$A' := P_k \left(\int_0^{\delta/N(0)} \frac{ie^{i\tau\sqrt{-\Delta}}}{\sqrt{-\Delta}} (|x|^{-3} * |u|^2)u(\tau) d\tau + \int_{\delta/N(0)}^{T_2} \frac{ie^{i\tau\sqrt{-\Delta}}}{\sqrt{-\Delta}} (1-\chi)\left(\frac{8x}{|\tau|}\right) (|x|^{-3} * |u|^2)u d\tau \right)$$

and

$$B' := P_k \left(\int_{\delta/N(0)}^{T_2} \frac{ie^{i\tau\sqrt{-\Delta}}}{\sqrt{-\Delta}} \chi\left(\frac{8x}{|\tau|}\right) (|x|^{-3} * |u|^2)u d\tau \right).$$

Then, I_2 can be expressed by

$$I_2 = \langle A + B, A' \rangle_{\dot{H}^{\frac{1}{2}}} + \langle A, A' + B' \rangle_{\dot{H}^{\frac{1}{2}}} - \langle A, A' \rangle_{\dot{H}^{\frac{1}{2}}} + \langle B, B' \rangle_{\dot{H}^{\frac{1}{2}}}.$$

First, we estimate the contribution of the term $\langle A + B, A' \rangle_{\dot{H}^{\frac{1}{2}}}$. Using Lemma 4.2 and (4.15), we get

$$\|A'\|_{\dot{H}^{\frac{1}{2}}} \lesssim \eta^2 \sum_{j \geq k-3} 2^{\frac{k-j}{4}} a_j + 2^{-\frac{k}{6}} b_k.$$

On the other hand, observing that

$$\lim_{T_1 \rightarrow T_-} \frac{1}{\sqrt{-\Delta}} \int_{T_1}^0 e^{it\sqrt{-\Delta}} (|x|^{-3} * |u|^2) u(t) dt \rightharpoonup v(0) \quad \text{in } \dot{H}^{\frac{1}{2}},$$

we obtain

$$\begin{aligned} & \lim_{T_1 \rightarrow T_-} \lim_{T_2 \rightarrow \infty} \left| \langle A + B, A' \rangle_{\dot{H}^{\frac{1}{2}}} \right| \\ &= \lim_{T_1 \rightarrow T_-} \lim_{T_2 \rightarrow \infty} \left\langle P_k \left(\int_{T_1}^0 \frac{e^{it\sqrt{-\Delta}}}{\sqrt{-\Delta}} (|x|^{-3} * |u|^2) u(t) dt \right), A' \right\rangle_{\dot{H}^{\frac{1}{2}}} \\ &= \langle P_k v(0), A'(+\infty) \rangle_{\dot{H}^{\frac{1}{2}}} \\ &\lesssim \|P_k v(0)\|_{\dot{H}^{\frac{1}{2}}} \|A'(+\infty)\|_{\dot{H}^{\frac{1}{2}}} \\ &\lesssim a_k(0) \left(\eta^2 \sum_{j \geq k-3} 2^{\frac{k-j}{4}} a_j + 2^{-\frac{k}{6}} b_k \right), \end{aligned} \tag{4.24}$$

where $A'(+\infty) = \lim_{T_2 \rightarrow +\infty} A'$. By symmetry, we also have

$$\lim_{T_1 \rightarrow T_-} \lim_{T_2 \rightarrow \infty} \left| \langle A, A' + B' \rangle_{\dot{H}^{\frac{1}{2}}} \right| \lesssim a_k(0) \left(\eta^2 \sum_{j \geq k-3} 2^{\frac{k-j}{4}} a_j + 2^{-\frac{k}{6}} b_k \right). \tag{4.25}$$

Next, we consider the contribution of the term $\langle A, A' \rangle_{\dot{H}^{\frac{1}{2}}}$. By (4.15) and (4.19), we get

$$\left| \langle A, A' \rangle_{\dot{H}^{\frac{1}{2}}} \right| \lesssim \|A\|_{\dot{H}^{\frac{1}{2}}} \|A'\|_{\dot{H}^{\frac{1}{2}}} \lesssim \left(\eta^2 \sum_{j \geq k-3} 2^{\frac{k-j}{4}} a_j \right)^2 + 2^{-\frac{k}{3}} b_k^2. \tag{4.26}$$

Finally, we turn to estimate the contribution of the term $\langle B, B' \rangle_{\dot{H}^{\frac{1}{2}}}$. We rewrite $\langle B, B' \rangle_{\dot{H}^{\frac{1}{2}}}$ as

$$\int_{T_1}^{-\frac{\delta}{N(0)}} \int_{\frac{\delta}{N(0)}}^{T_2} \left\langle \chi \left(\frac{8x}{|t|} \right) (|x|^{-3} * |u|^2) u, P_k^2 \left(\frac{e^{i(\tau-t)\sqrt{-\Delta}}}{(-\Delta)^{\frac{1}{2}}} \chi \left(\frac{8x}{|\tau|} \right) (|x|^{-3} * |u|^2) u \right) \right\rangle_{L_x^2} dt d\tau.$$

The kernel of $P_k^2 \frac{e^{i(\tau-t)\sqrt{-\Delta}}}{(-\Delta)^{\frac{1}{2}}}$ is given by

$$\begin{aligned} K_k(x) &= K_k(|x|) = c \int_{\mathbb{R}^d} e^{ix \cdot \xi} \phi\left(\frac{|\xi|}{2^k}\right)^2 e^{i(\tau-t)|\xi|} |\xi|^{-1} d\xi \\ &= c \int_0^\infty \phi\left(\frac{\rho}{2^k}\right)^2 e^{i(\tau-t)\rho} \rho^{-1} \rho^{d-1} \int_{\mathbb{S}^{d-1}} e^{i\rho|x|\theta \cdot w} d\theta d\rho \\ &= c 2^{k(d-1)} \int_{\mathbb{S}^{d-1}} \int_0^\infty \phi(\rho)^2 e^{i2^k \rho [(\tau-t)+|x|\theta \cdot w]} \rho^{d-2} d\rho d\theta, \end{aligned}$$

where $x = |x|w$, $\xi = \rho\theta$, $w, \theta \in \mathbb{S}^{d-1}$. By the stationary phase argument, we obtain for $L \in \mathbb{N}$

$$\begin{aligned} |K_k(x-y)| &\lesssim \frac{2^{(d-1)k}}{\langle 2^k |(\tau-t) - |x-y|| \rangle^L} \\ &\lesssim 2^{(d-1)k} \langle 2^k |\tau-t| \rangle^{-L} \\ &\lesssim 2^{(d-1)k} \langle 2^k |\tau| \rangle^{-L/2} \langle 2^k |t| \rangle^{-L/2}, \end{aligned} \tag{4.27}$$

where we have applied the support property

$$|x| \leq \frac{|t|}{4}, \quad |y| \leq \frac{|\tau|}{4}, \quad \text{and} \quad \tau > \frac{\delta}{N(0)}, \quad t < -\frac{\delta}{N(0)}$$

to get

$$|x-y| \leq \frac{|t-\tau|}{4} \implies (\tau-t) - |x-y| \geq \frac{1}{2}|\tau-t|,$$

and

$$\langle 2^k |\tau-t| \rangle^2 \geq \langle 2^k |\tau| \rangle \langle 2^k |t| \rangle.$$

By the Hölder inequality, the Hardy-Littlewood-Sobolev inequality and the Sobolev embedding, we estimate by choosing $L = d$

$$\begin{aligned}
& \langle B, B' \rangle_{\dot{H}^{\frac{1}{2}}} \\
& \lesssim \int_{T_1}^{-\frac{\delta}{N(0)}} \int_{\frac{\delta}{N(0)}}^{T_2} \left| \left\langle \chi\left(\frac{8x}{|t|}\right) (|x|^{-3} * |u|^2) u, K_k * [(|x|^{-3} * |u|^2) \chi\left(\frac{8x}{|\tau|}\right) u] \right\rangle_{L^2} \right| dt d\tau \\
& \lesssim \int_{T_1}^{-\frac{\delta}{N(0)}} \int_{\frac{\delta}{N(0)}}^{T_2} \|(|x|^{-3} * |u|^2) \chi\left(\frac{8x}{|t|}\right) u\|_{L_x^1} \|(|x|^{-3} * |u|^2) \chi\left(\frac{8x}{|\tau|}\right) u\|_{L_x^1} 2^{(d-1)k} \langle 2^k |\tau - t| \rangle^{-d} dt d\tau \\
& \lesssim \| |x|^{-3} * |u|^2 \|_{L_t^\infty L_x^{\frac{d}{2}}}^2 \|u\|_{L_x^{\frac{2d}{d-1}}}^2 \int_{-\infty}^{-\frac{\delta}{N(0)}} \int_{\frac{\delta}{N(0)}}^\infty \left\| \chi\left(\frac{8x}{|t|}\right) \right\|_{L_x^{\frac{2d}{d-3}}} \left\| \chi\left(\frac{8x}{|\tau|}\right) \right\|_{L_x^{\frac{2d}{d-3}}} 2^{(d-1)k} \langle 2^k |\tau - t| \rangle^{-d} dt d\tau \\
& \lesssim \|u\|_{L_t^\infty \dot{H}_x^{\frac{1}{2}}}^6 \int_{-\infty}^{-\frac{\delta}{N(0)}} \int_{\frac{\delta}{N(0)}}^\infty (2^k |t|)^{\frac{d-3}{2}} (2^k |\tau|)^{\frac{d-3}{2}} 2^{2k} \langle 2^k |\tau| \rangle^{-d/2} \langle 2^k |t| \rangle^{-d/2} dt d\tau \\
& \lesssim \int_{-\infty}^{-2^k \frac{\delta}{N(0)}} \int_{2^k \frac{\delta}{N(0)}}^\infty |t|^{\frac{d-3}{2}} |\tau|^{\frac{d-3}{2}} \langle |\tau| \rangle^{-d/2} \langle |t| \rangle^{-d/2} dt d\tau \\
& \lesssim \left(\int_{2^k \frac{\delta}{N(0)}}^\infty \langle |t| \rangle^{-3/2} dt \right)^2 \lesssim \min \left\{ 2^{-\frac{k}{2}} N(0)^{\frac{1}{2}}, 1 \right\}. \tag{4.28}
\end{aligned}$$

Thus, collecting (4.22), (4.23), (4.24), (4.26) and (4.28) yields that

$$a_k(0)^2 \lesssim a_k(0) \left(\eta^2 \sum_{j \geq k-3} 2^{\frac{k-j}{4}} a_j + 2^{-\frac{k}{6}} b_k \right) + \left(\eta^2 \sum_{j \geq k-3} 2^{\frac{k-j}{4}} a_j \right)^2 + 2^{-\frac{k}{3}} b_k^2 + \min \left\{ 2^{-\frac{k}{2}} N(0)^{\frac{1}{2}}, 1 \right\}.$$

And so we obtain by Young's inequality

$$a_k(0) \lesssim \eta^2 \sum_{j \geq k-3} 2^{\frac{k-j}{4}} a_j + 2^{-\frac{k}{6}} b_k + \min \left\{ 2^{-\frac{k}{4}} N(0)^{\frac{1}{4}}, 1 \right\}.$$

Combining this inequality with (4.13), we obtain

$$\begin{aligned}
\alpha_k(0) &= \sum_j 2^{-\frac{|j-k|}{5}} a_j(0) \\
&\lesssim \eta^2 \sum_j 2^{-\frac{|j-k|}{5}} \sum_{j_1 \geq j-3} 2^{\frac{j-j_1}{4}} a_{j_1} + \sum_j 2^{-\frac{|j-k|}{5}} 2^{-\frac{j}{6}} b_j + \sum_j 2^{-\frac{|j-k|}{5}} \min \left\{ 2^{-\frac{j}{4}} N(0)^{\frac{1}{4}}, 1 \right\} \\
&\lesssim \eta^2 \alpha_k + \sum_j 2^{-\frac{|j-k|}{5}} 2^{-\frac{j}{6}} b_j + \sum_j 2^{-\frac{|j-k|}{5}} 2^{-\frac{j}{6}} \min \left\{ 2^{-\frac{1}{12}j} N(0)^{\frac{1}{4}}, 2^{\frac{j}{6}} \right\}.
\end{aligned}$$

Choosing η small and using (4.12), we deduce that

$$\alpha_k(0) \lesssim \sum_j 2^{-\frac{|j-k|}{5}} 2^{-\frac{j}{6}} b_j + \sum_j 2^{-\frac{|j-k|}{5}} 2^{-\frac{j}{6}} c_j, \tag{4.29}$$

where $c_j := \min \left\{ 2^{-\frac{1}{12}j} N(0)^{\frac{1}{4}}, 2^{\frac{j}{6}} \right\}$ obeys

$$\| \{c_j\}_{j \in \mathbb{Z}} \|_{l^2} \lesssim N(0)^{\frac{1}{6}}. \tag{4.30}$$

Using Schur's test, (4.29), (4.19) and (4.30), we obtain

$$\left\| \left\{ 2^{\frac{k}{6}} \alpha_k(0) \right\}_{k \in \mathbb{Z}} \right\|_{l_2} \lesssim N(0)^{\frac{1}{6}}. \quad (4.31)$$

This together with (4.10) ends the proof of Theorem 4.1.

Now we use Theorem 4.1 to prove Theorem 1.5.

• **Proof of Theorem 1.5.** We first establish a refined Strichartz estimate in a small time interval.

Lemma 4.3 (Refined Strichartz estimate). *Let $u : (T_-, +\infty) \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a radial solution to problem (1.1) which is almost periodic modulo symmetries in the sense of Theorem 1.4. Then, there exists a $\delta > 0$ sufficiently small such that for any $t_0 \in \mathbb{R}^+$,*

$$\|u\|_{L_t^q L_x^r(J_\delta(t_0) \times \mathbb{R}^d)} \lesssim N(t_0)^{\frac{1}{6}}, \quad \frac{1}{q} \leq \frac{d-1}{2} \left(\frac{1}{2} - \frac{1}{r} \right), \quad \frac{1}{q} + \frac{d}{r} = \frac{d}{2} - \frac{2}{3}, \quad (4.32)$$

where $J_\delta(t_0) := [t_0 - \frac{\delta}{N(t_0)}, t_0 + \frac{\delta}{N(t_0)}]$. In particular,

$$\|u\|_{L_t^3 L_x^{\frac{2d}{d-2}}(J_\delta(t_0) \times \mathbb{R}^d)} \lesssim N(t_0)^{\frac{1}{6}}. \quad (4.33)$$

Proof. Let $Y := L_t^\infty L_x^{\frac{6d}{3d-4}} \cap L_t^3 L_x^{\frac{2d}{d-2}}(J_\delta(t_0) \times \mathbb{R}^d)$. We use the Strichartz estimate to get

$$\begin{aligned} \|u\|_Y &\lesssim \|\vec{u}(t_0)\|_{\dot{H}^{\frac{2}{3}} \times \dot{H}^{-\frac{1}{3}}} + \|(|x|^{-3} * |u|^2)u\|_{L_t^1 L_x^{\frac{6d}{3d+2}}(J_\delta(t_0) \times \mathbb{R}^d)} \\ &\lesssim N(t_0)^{\frac{1}{6}} + \left(\frac{\delta}{N(t_0)} \right)^{\frac{1}{3}} \|u\|_{L_t^{\frac{9}{2}} L_x^{\frac{18d}{9d-16}}(J_\delta(t_0) \times \mathbb{R}^d)}^3 \\ &\lesssim N(t_0)^{\frac{1}{6}} + \left(\frac{\delta}{N(t_0)} \right)^{\frac{1}{3}} \|u\|_Y^3. \end{aligned}$$

This implies that

$$N(t_0)^{-\frac{1}{6}} \|u\|_Y \lesssim 1 + \delta^{\frac{1}{3}} (N(t_0)^{-\frac{1}{6}} \|u\|_Y)^3.$$

Therefore, we obtain by choosing $\delta > 0$ sufficiently small

$$\|u\|_Y \lesssim N(t_0)^{\frac{1}{6}},$$

as desired. \square

As a consequence of (4.33) and the Hardy-Littlewood-Sobolev inequality, we have the following estimate.

Corollary 4.1. *Let u be as in Lemma 4.3. Then, we have*

$$\|(|x|^{-3} * |u|^2)u\|_{L_t^1 L_x^2(J_\delta(t_0) \times \mathbb{R}^d)} \lesssim N(t_0)^{\frac{1}{2}}. \quad (4.34)$$

Now, we turn to prove Theorem 1.5. By time translation and recalling $v(t) = u(t) + \frac{i}{\sqrt{-\Delta}} u_t$, we easily see that (1.21) can be reduced to showing

$$\|P_{\leq k} v(0)\|_{\dot{H}^1} \lesssim \begin{cases} N(0)^{\frac{1}{2}} & \text{if } d \geq 5 \\ N(0)^{\frac{1}{3}} & \text{if } d = 4 \end{cases} \quad (4.35)$$

uniformly for all $k \geq k_0 > 0$.

By Duhamel formula, we have for $T_- < T_1 < 0 < T_2 < +\infty$

$$\begin{aligned}
& \|P_{\leq k} v(0)\|_{\dot{H}^1}^2 = \langle P_{\leq k} v(0), P_{\leq k} v(0) \rangle_{\dot{H}^1} \\
& = \left\langle P_{\leq k} \left(e^{iT_2 \sqrt{-\Delta}} v(T_2) \mp \frac{i}{\sqrt{-\Delta}} \int_0^{T_2} e^{it\sqrt{-\Delta}} (|x|^{-3} * |u|^2) u(t) dt \right), \right. \\
& \quad \left. P_{\leq k} \left(e^{iT_1 \sqrt{-\Delta}} v(T_1) \pm \frac{i}{\sqrt{-\Delta}} \int_{T_1}^0 e^{i\tau \sqrt{-\Delta}} (|x|^{-3} * |u|^2) u(\tau) d\tau \right) \right\rangle_{\dot{H}^1} \\
& = \left\langle P_{\leq k} \left(\int_0^{T_2} e^{it\sqrt{-\Delta}} (|x|^{-3} * |u|^2) u(t) dt \right), P_{\leq k} \left(i \int_{T_1}^0 e^{i\tau \sqrt{-\Delta}} (|x|^{-3} * |u|^2) u(\tau) d\tau \right) \right\rangle_{L^2} \\
& \quad + \left\langle e^{iT_2 \sqrt{-\Delta}} P_{\leq k} v(T_2), e^{iT_1 \sqrt{-\Delta}} P_{\leq k} v(T_1) \right\rangle_{\dot{H}^1} \\
& \quad + \left\langle P_{\leq k} \left(\int_0^{T_2} e^{it\sqrt{-\Delta}} (|x|^{-3} * |u|^2) u(t) dt \right), e^{iT_1 \sqrt{-\Delta}} P_{\leq k} v(T_1) \right\rangle_{\dot{H}^1} \\
& \quad + \left\langle e^{iT_2 \sqrt{-\Delta}} P_{\leq k} v(T_2), \pm P_{\leq k} \left(\frac{i}{\sqrt{-\Delta}} \int_{T_1}^0 e^{i\tau \sqrt{-\Delta}} (|x|^{-3} * |u|^2) u(\tau) d\tau \right) \right\rangle_{\dot{H}^1} \\
& =: I_1 + I_2 + I_3 + I_4, \tag{4.36}
\end{aligned}$$

where $\langle f, g \rangle_{\dot{H}^1} := \int_{\mathbb{R}^d} \sqrt{-\Delta} f \cdot \sqrt{-\Delta} g dx$.

First, we claim that

$$\lim_{T_2 \nearrow \infty} I_2 = \lim_{T_2 \nearrow \infty} I_4 = 0 \quad \text{and} \quad \lim_{T_1 \searrow T_-} \lim_{T_2 \nearrow \infty} I_3 = 0. \tag{4.37}$$

Since $\sqrt{-\Delta} P_{\leq k} v(T_1) \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^d)$ for fixed T_1 and k , we have by the no-waste Duhamel formula

$$\lim_{T_2 \nearrow \infty} I_2 = 0. \tag{4.38}$$

On the other hand, it follows from Corollary 4.1 that

$$\int_{-\frac{\delta}{N(0)}}^{\frac{\delta}{N(0)}} \|P_{\leq k} ((|x|^{-3} * |u|^2) u)\|_{L_x^2} dt \lesssim N(0)^{\frac{1}{2}}$$

and

$$\|(|x|^{-3} * |u|^2) u\|_{L_t^1 L_x^2([T_1, 0] \times \mathbb{R}^d)} < \infty, \quad T_1 > T_-.$$

Hence,

$$P_{\leq k} \int_{T_1}^0 e^{i\tau \sqrt{-\Delta}} (|x|^{-3} * |u|^2) u(\tau) d\tau \in \dot{H}^{\frac{1}{2}}.$$

Using the no-waste Duhamel formula again, we obtain

$$\lim_{T_2 \nearrow \infty} I_4 = \lim_{T_2 \nearrow \infty} \left\langle e^{iT_2 \sqrt{-\Delta}} P_{\leq k} v(T_2), P_{\leq k} \int_{T_1}^0 e^{i\tau \sqrt{-\Delta}} (|x|^{-3} * |u|^2) u(\tau) d\tau \right\rangle_{\dot{H}^{\frac{1}{2}}} = 0.$$

Observing that

$$P_{\leq k} \left(\int_0^{T_2} e^{i\tau \sqrt{-\Delta}} (|x|^{-3} * |u|^2) u(\tau) d\tau \right) = \mp i \sqrt{-\Delta} P_{\leq k} v(0) \pm i \sqrt{-\Delta} e^{iT_2 \sqrt{-\Delta}} P_{\leq k} v(T_2),$$

one easily deduces that

$$\begin{aligned} \lim_{T_1 \searrow T_-} \lim_{T_2 \nearrow \infty} I_3 &= \lim_{T_1 \searrow T_-} \lim_{T_2 \nearrow \infty} \left\langle e^{iT_1 \sqrt{-\Delta}} P_{\leq k} v(T_1), \sqrt{-\Delta} P_{\leq k} v(0) - e^{iT_2 \sqrt{-\Delta}} P_{\leq k} v(T_2) \right\rangle_{\dot{H}^1} \\ &= \lim_{T_1 \searrow T_-} \left\langle e^{iT_1 \sqrt{-\Delta}} P_{\leq k} v(T_1), \sqrt{-\Delta} P_{\leq k} v(0) \right\rangle_{\dot{H}^1} \\ &= 0. \end{aligned}$$

And so the claim (4.37) follows.

It remains to estimate the contribution of the term I_1 . We claim

$$\left| \lim_{T_1 \searrow T_-} \lim_{T_2 \nearrow T_+} I_1 \right| \lesssim \begin{cases} \|P_{\leq k} v(0)\|_{\dot{H}^1} N(0)^{\frac{1}{2}} + N(0) & \text{if } d \geq 5, \\ \|P_{\leq k} v(0)\|_{\dot{H}^1} N(0)^{\frac{2}{3}} + N(0)^{\frac{2}{3}} & \text{if } d = 4. \end{cases} \quad (4.39)$$

Assuming that this claim holds for moment, then we have by (4.37)

$$\|P_{\leq k} v(0)\|_{\dot{H}^1}^2 \lesssim \begin{cases} \|P_{\leq k} v(0)\|_{\dot{H}^1} N(0)^{\frac{1}{2}} + N(0) & \text{if } d \geq 5, \\ \|P_{\leq k} v(0)\|_{\dot{H}^1} N(0)^{\frac{1}{3}} + N(0)^{\frac{2}{3}} & \text{if } d = 4. \end{cases} \quad (4.40)$$

Hence by Cauchy's inequality with ϵ , we have

$$\|P_{\leq k} v(0)\|_{\dot{H}^1} \lesssim \begin{cases} N(0)^{\frac{1}{2}} & \text{if } d \geq 5, \\ N(0)^{\frac{1}{3}} & \text{if } d = 4. \end{cases} \quad (4.41)$$

This implies (4.35). Thus, Theorem 1.5 follows.

• **The proof of claim (4.39).** We first use Corollary 4.1 to get the small time estimate

$$\int_{-\frac{\delta}{N(0)}}^{\frac{\delta}{N(0)}} \|P_{\leq k}(|x|^{-3} * |u|^2)u\|_{L_x^2} dt \lesssim N(0)^{\frac{1}{2}}. \quad (4.42)$$

By the Hölder inequality, the Hardy-Littlewood-Sobolev inequality, the Sobolev and radial Sobolev embedding, we have for $d \geq 5$

$$\||x|^{\frac{3}{2}}(|x|^{-3} * |u|^2)u\|_{L_x^2(\mathbb{R}^d)} \lesssim \||x|^{\frac{3}{2}}u\|_{L_x^{\frac{2d}{d-4}}} \|u\|_{L_x^{\frac{2d}{d-1}}}^2 \lesssim \|u\|_{\dot{H}_x^{\frac{1}{2}}(\mathbb{R}^d)}^3,$$

and for $d = 4$

$$\||x|^{\frac{4}{3}}(|x|^{-3} * |u|^2)u\|_{L_x^2} \lesssim \||x|^{\frac{4}{3}}u\|_{L_x^\infty(\mathbb{R}^4)} \|u\|_{L_x^{\frac{8}{3}}(\mathbb{R}^4)}^2 \leq \|u\|_{\dot{H}_x^{\frac{2}{3}}(\mathbb{R}^4)} \|u\|_{\dot{H}_x^{\frac{1}{2}}(\mathbb{R}^4)}^2.$$

This implies that

$$\|(1 - \chi)\left(\frac{8x}{|t|}\right)(|x|^{-3} * |u|^2)u\|_{L_x^2(\mathbb{R}^d)} \lesssim \begin{cases} \frac{1}{|t|^{\frac{3}{2}}} \|u\|_{\dot{H}^{\frac{1}{2}}}^3 & \text{if } d \geq 5, \\ \frac{1}{|t|^{\frac{4}{3}}} \|u\|_{\dot{H}^{\frac{2}{3}}} \|u\|_{\dot{H}^{\frac{1}{2}}}^2 & \text{if } d = 4. \end{cases}$$

Since $\|u(t, x)\|_{\dot{H}_x^{\frac{2}{3}}(\mathbb{R}^d)} \lesssim N(t)^{\frac{1}{6}} \lesssim 1$ for $t \in \mathbb{R}^+$ by Theorem 4.1 and $N(t) \leq 1$ when $t \in \mathbb{R}^+$, we have the large time estimate

$$\begin{aligned} & \left\| P_{\leq k} \left(\int_{\delta/N(0)}^{\infty} e^{it\sqrt{-\Delta}} (1 - \chi) \left(\frac{8x}{|t|} \right) (|x|^{-3} * |u|^2) u \, dt \right) \right\|_{L_x^2} \\ & \lesssim \int_{\delta/N(0)}^{\infty} \left\| (1 - \chi) \left(\frac{8x}{|t|} \right) (|x|^{-3} * |u|^2) u \right\|_{L_x^2} \, dt \\ & \lesssim \begin{cases} \delta^{-\frac{1}{2}} N(0)^{\frac{1}{2}} & \text{if } d \geq 5, \\ \delta^{-\frac{1}{3}} N(0)^{\frac{1}{3}} & \text{if } d = 4. \end{cases} \end{aligned} \quad (4.43)$$

Define

$$A := P_{\leq k} \left(\int_0^{\delta/N(0)} e^{it\sqrt{-\Delta}} (|x|^{-3} * |u|^2) u(t) \, dt + \int_{\delta/N(0)}^{T_2} e^{it\sqrt{-\Delta}} (1 - \chi) \left(\frac{8x}{|t|} \right) (|x|^{-3} * |u|^2) u \, dt \right)$$

and

$$B := P_{\leq k} \left(\int_{\delta/N(0)}^{T_2} e^{it\sqrt{-\Delta}} \chi \left(\frac{8x}{|t|} \right) (|x|^{-3} * |u|^2) u \, dt \right).$$

Similarly,

$$A' := P_{\leq k} \left(\int_{-\delta/N(0)}^0 e^{i\tau\sqrt{-\Delta}} (|x|^{-3} * |u|^2) u(\tau) \, d\tau + \int_{T_1}^{-\delta/N(0)} e^{i\tau\sqrt{-\Delta}} (1 - \chi) \left(\frac{8x}{|\tau|} \right) (|x|^{-3} * |u|^2) u \, d\tau \right)$$

and

$$B' := P_{\leq k} \left(\int_{T_1}^{-\delta/N(0)} e^{i\tau\sqrt{-\Delta}} \chi \left(\frac{8x}{|\tau|} \right) (|x|^{-3} * |u|^2) u \, d\tau \right).$$

Hence, we have

$$I_1 = \langle A + B, A' \rangle_{L^2} + \langle A, A' + B' \rangle_{L^2} + \langle B, B' \rangle_{L^2} - \langle A, A' \rangle_{L^2}. \quad (4.44)$$

First, we estimate the contribution of the term $\langle A, A' \rangle_{L^2}$. By (4.42) and (4.43), we estimate

$$\|A\|_{L_x^2} + \|A'\|_{L_x^2} \lesssim \begin{cases} N(0)^{\frac{1}{2}} & \text{if } d \geq 5 \\ N(0)^{\frac{1}{3}} & \text{if } d = 4, \end{cases} \implies \langle A, A' \rangle \lesssim \begin{cases} N(0) & \text{if } d \geq 5 \\ N(0)^{\frac{2}{3}} & \text{if } d = 4. \end{cases} \quad (4.45)$$

Next, we estimate the contribution of the term $\langle B, B' \rangle_{L^2}$. We rewrite $\langle B, B' \rangle_{L^2}$ as

$$\begin{aligned} & \langle B, B' \rangle_{L^2} \\ &= \left\langle P_{\leq k} \left(\int_{\frac{\delta}{N(0)}}^{T_2} e^{it\sqrt{-\Delta}} \chi \left(\frac{8x}{|t|} \right) (|x|^{-3} * |u|^2) u \, dt \right), \right. \\ & \quad \left. P_{\leq k} \left(\int_{T_1}^{-\frac{\delta}{N(0)}} e^{i\tau\sqrt{-\Delta}} \chi \left(\frac{8x}{|\tau|} \right) (|x|^{-3} * |u|^2) u \, d\tau \right) \right\rangle_{L^2} \\ &= \int_{T_1}^{-\frac{\delta}{N(0)}} \int_{\frac{\delta}{N(0)}}^{T_2} \left\langle \chi \left(\frac{8x}{|t|} \right) (|x|^{-3} * |u|^2) u, P_{\leq k}^2 \left(e^{i(\tau-t)\sqrt{-\Delta}} \chi \left(\frac{8x}{|\tau|} \right) (|x|^{-3} * |u|^2) u \right) \right\rangle_{L_x^2} \, dt \, d\tau. \end{aligned}$$

In the same way as deriving in (4.27), we estimate the kernel of $P_{\leq k}^2 e^{i(\tau-t)\sqrt{-\Delta}}$ as follows

$$|\tilde{K}_k(x)| \lesssim 2^{dk} \langle 2^k |\tau| \rangle^{-d/2} \langle 2^k |t| \rangle^{-d/2}.$$

Combining this inequality with the proof process of (4.28), we obtain for $2^k \gg N(0)$

$$\langle B, B' \rangle_{L^2} \lesssim 2^k \left(\int_{2^k \frac{\delta}{N(0)}}^{\infty} \langle |t| \rangle^{-3/2} dt \right)^2 \lesssim N(0). \quad (4.46)$$

Finally, we only need to estimate the contribution of $\langle A, A' + B' \rangle_{L^2}$, since $\langle A + B, A' \rangle_{L^2}$ has the same estimate by symmetry. Observing that

$$P_{\leq k} \left(\int_{T_1}^0 e^{i\tau\sqrt{-\Delta}} (|x|^{-3} * |u|^2) u(\tau) d\tau \right) = \mp i\sqrt{-\Delta} P_{\leq k} v(0) \pm i\sqrt{-\Delta} e^{iT_1\sqrt{-\Delta}} P_{\leq k} v(T_1),$$

we obtain

$$\langle A, A' + B' \rangle_{L^2} = \left\langle A, \mp i\sqrt{-\Delta} P_{\leq k} v(0) \pm i\sqrt{-\Delta} e^{iT_1\sqrt{-\Delta}} P_{\leq k} v(T_1) \right\rangle_{L^2}. \quad (4.47)$$

Using (4.45), we have

$$\left| \left\langle A, \sqrt{-\Delta} P_{\leq k} v(0) \right\rangle_{L_x^2} \right| \leq \|A\|_{L_x^2} \|\sqrt{-\Delta} P_{\leq k} v(0)\|_{L_x^2} \lesssim \begin{cases} \|P_{\leq k} v(0)\|_{\dot{H}^1} N(0)^{\frac{1}{2}} & \text{if } d \geq 5 \\ \|P_{\leq k} v(0)\|_{\dot{H}^1} N(0)^{\frac{1}{3}} & \text{if } d = 4. \end{cases} \quad (4.48)$$

We claim that

$$\lim_{T_1 \searrow T_-} \lim_{T_2 \nearrow \infty} \left| \left\langle A, \sqrt{-\Delta} e^{iT_1\sqrt{-\Delta}} P_{\leq k} v(T_1) \right\rangle_{L_x^2} \right| = 0. \quad (4.49)$$

In fact, since $\|(-\Delta)^{\frac{1}{4}} A\|_{L_x^2} \lesssim 2^{\frac{k}{2}} N(0)^{\frac{1}{3}}$ by (4.45), we deduce that for a fixed k and letting $T_2 \nearrow \infty$, $(-\Delta)^{\frac{1}{4}} A$ converges to

$$(-\Delta)^{\frac{1}{4}} P_{\leq k} \left(\int_0^{\frac{\delta}{N(0)}} e^{it\sqrt{-\Delta}} (|x|^{-3} * |u|^2) u(t) dt + \int_{\frac{\delta}{N(0)}}^{\infty} e^{it\sqrt{-\Delta}} (1 - \chi) \left(\frac{8x}{|t|} \right) (|x|^{-3} * |u|^2) u dt \right),$$

in $L_x^2(\mathbb{R}^d)$. Then, we obtain by the no-waste Duhamel formula

$$\lim_{T_1 \searrow T_-} \lim_{T_2 \nearrow T_+} \left\langle A, \sqrt{-\Delta} e^{iT_1\sqrt{-\Delta}} P_{\leq k} v(T_1) \right\rangle_{L_x^2} = 0, \quad (4.50)$$

as desired.

Collecting (4.44)-(4.46) and (4.50), we obtain (4.40). Therefore, we conclude Theorem 1.5.

4.2. Proof of Proposition 1.1. First, Theorem 1.5 implies that

$$\|(u, u_t)(t)\|_{L_t^\infty(\mathbb{R}, \dot{H}^1 \times L^2)} \lesssim 1.$$

Hence, by the same argument in spirit of Lemma 6.12 in [27], Proposition 1.1 can be reduced to show that

$$\|(u, u_t)(t)\|_{L_t^\infty(\mathbb{R}, \dot{H}^2 \times \dot{H}^1)} \lesssim 1. \quad (4.51)$$

Let $v(t)$ be defined as in (4.21). Then, (4.51) can be reduced to show that

$$\|P_{\leq k} v(0)\|_{\dot{H}^2} \lesssim 1 \quad (4.52)$$

uniformly for all $k > k_0$. The proof is the same as in the proof of Theorem 1.5. Employing the No-waste Duhamel formula as in the proof of Theorem 1.5, we only need to prove the following double Duhamel term obeying

$$\left| \left\langle P_{\leq k} \left(\int_{T_1}^0 e^{it\sqrt{-\Delta}} \nabla (|x|^{-3} * |u|^2) u \right) (t) dt, P_{\leq k} \left(\int_0^{T_2} e^{i\tau\sqrt{-\Delta}} \nabla (|x|^{-3} * |u|^2) u \right) (\tau) d\tau \right\rangle_{L_x^2} \right| \lesssim 1$$

uniformly for $T_1 < 0 < T_2$. To do this, we suffice to show the similar estimate as in (4.42), (4.43) and (4.46).

For this purpose, we first establish the following refined Strichartz estimate in a small time interval.

Lemma 4.4. *Let $u : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a radial solution to problem (1.1) which is almost periodic modulo symmetries in the sense of Theorem 1.4 with $N(t) \equiv 1$. Then, there exists a $\delta > 0$ such that for any $t_0 \in \mathbb{R}$ and for $J := (t_0 - \delta, t_0 + \delta)$, we have*

$$\|u\|_{L_t^2 L_x^{\frac{2d}{d-3}}(J \times \mathbb{R}^d)} \lesssim 1. \quad (4.53)$$

Proof. Without loss of generality, we assume that $t_0 = 0$. Let

$$Z(J) = L_t^\infty(J, \dot{H}^1) \cap L_t^2(J, L_x^{\frac{2d}{d-3}}).$$

We have by Strichartz estimate and Theorem 1.5

$$\begin{aligned} \|u\|_{Z(J)} &\lesssim \|(u_0, u_1)\|_{\dot{H}^1 \times L^2} + \|(|x|^{-3} * |u|^2)u\|_{L_t^1 L_x^2(J \times \mathbb{R}^d)} \\ &\lesssim \|(u_0, u_1)\|_{\dot{H}^1 \times L^2} + \|u\|_{L_t^3 L_x^{\frac{2d}{d-2}}(J \times \mathbb{R}^d)}^3 \\ &\lesssim \|(u_0, u_1)\|_{\dot{H}^1 \times L^2} + \delta \|u\|_{L_t^\infty(J, \dot{H}^1)}^3 \\ &\lesssim 1 + \delta \|u\|_{Z(J)}^3. \end{aligned}$$

This ends the proof by choosing $\delta > 0$ sufficiently small. \square

Using Lemma 4.4, we have the small time estimate

$$\begin{aligned} &\left\| P_{\leq k} \left(\int_0^\delta e^{i\tau\sqrt{-\Delta}} \nabla (|x|^{-3} * |u|^2) u \right) (\tau) d\tau \right\|_{L_x^2} \\ &\lesssim \int_0^\delta \|\nabla (|x|^{-3} * |u|^2) u\|_{L_x^2} d\tau \lesssim \|\nabla u\|_{L_t^\infty L_x^2} \|u\|_{L_t^2 L_x^{\frac{2d}{d-3}}([0, \delta])}^2 \lesssim 1. \end{aligned} \quad (4.54)$$

On the other hand, when $d \geq 5$, by the radial Sobolev embedding: $\| |x|^{\frac{3}{2}} u \|_{L_x^{\frac{2d}{d-4}}(\mathbb{R}^d)} \lesssim \|u\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^d)}$, the Hölder inequality and the Hardy-Littlewood-Sobolev inequality, we obtain

$$\begin{aligned} \left\| (1 - \chi) \left(\frac{8x}{|t|} \right) (|x|^{-3} * (\nabla |u|^2)) u \right\|_{L_x^2} &\lesssim \left\| (1 - \chi) \left(\frac{8x}{|t|} \right) u \right\|_{L_x^{\frac{2d}{d-4}}} \| |x|^{-3} * (u \nabla u) \|_{L_x^{\frac{d}{2}}} \\ &\lesssim |t|^{-\frac{3}{2}} \| |x|^{\frac{3}{2}} u \|_{L_x^{\frac{2d}{d-4}}} \|\nabla u\|_{L_x^2} \|u\|_{L_x^{\frac{2d}{d-2}}} \\ &\lesssim |t|^{-\frac{3}{2}}. \end{aligned}$$

When $d = 4$, by the radial Sobolev embedding: $\| |x|^{\frac{4}{3}} u \|_{L^\infty(\mathbb{R}^4)} \lesssim \| u \|_{\dot{H}^{\frac{2}{3}}(\mathbb{R}^4)}$, we obtain

$$\left\| (1 - \chi) \left(\frac{8x}{|t|} \right) (|x|^{-3} * (\nabla |u|^2)) u \right\|_{L_x^2} \lesssim \left\| (1 - \chi) \left(\frac{8x}{|t|} \right) u \right\|_{L_x^\infty} \| |x|^{-3} * (u \nabla u) \|_{L_x^2} \lesssim |t|^{-\frac{4}{3}}.$$

Similarly, when $d \geq 5$, using the radial Sobolev embedding: $\| |x|^{\frac{3}{2}} f \|_{L_x^\infty(\mathbb{R}^d)} \lesssim \| |\nabla|^{\frac{1}{2}} f \|_{L_x^{\frac{d}{2}}(\mathbb{R}^d)}$, the Hölder inequality and the Hardy-Littlewood-Sobolev inequality, we get

$$\begin{aligned} \left\| (1 - \chi) \left(\frac{8x}{|t|} \right) (|x|^{-3} * (|u|^2)) \nabla u \right\|_{L_x^2(\mathbb{R}^d)} &\lesssim \| \nabla u \|_{L_x^2} \left\| (1 - \chi) \left(\frac{8x}{|t|} \right) (|x|^{-3} * (|u|^2)) \right\|_{L_x^\infty(\mathbb{R}^d)} \\ &\lesssim |t|^{-\frac{3}{2}} \left\| |x|^{\frac{3}{2}} (|x|^{-3} * (|u|^2)) \right\|_{L_x^\infty(\mathbb{R}^d)} \\ &\lesssim |t|^{-\frac{3}{2}} \left\| |\nabla|^{\frac{1}{2}} (|x|^{-3} * |u|^2) \right\|_{L_x^{\frac{d}{2}}(\mathbb{R}^d)} \\ &\lesssim |t|^{-\frac{3}{2}} \| \nabla u \|_{L_x^2(\mathbb{R}^d)} \| |\nabla|^{\frac{1}{2}} u \|_{L_x^2(\mathbb{R}^d)}, \end{aligned}$$

and when $d = 4$, by $\| |x|^{\frac{4}{3}} u \|_{L_x^\infty(\mathbb{R}^4)} \lesssim \| u \|_{\dot{H}^{\frac{2}{3}}(\mathbb{R}^4)}$, we have

$$\left\| (1 - \chi) \left(\frac{8x}{|t|} \right) (|x|^{-3} * (|u|^2)) \nabla u \right\|_{L_x^2(\mathbb{R}^4)} \lesssim |t|^{-\frac{4}{3}}.$$

Hence, we have the large time estimate

$$\begin{aligned} &\left\| P_{\leq k} \left(\int_\delta^{T_2} e^{i\tau\sqrt{-\Delta}} (1 - \chi) \left(\frac{8x}{|t|} \right) \nabla (|x|^{-3} * |u|^2) u(\tau) d\tau \right) \right\|_{L_x^2(\mathbb{R}^d)} \\ &\lesssim \int_\delta^\infty \left\| (1 - \chi) \left(\frac{8x}{|t|} \right) \nabla (|x|^{-3} * |u|^2) u \right\|_{L_x^2(\mathbb{R}^d)} d\tau \\ &\lesssim \begin{cases} \delta^{-\frac{1}{2}} & \text{if } d \geq 5 \\ \delta^{-\frac{1}{3}} & \text{if } d = 4. \end{cases} \end{aligned} \tag{4.55}$$

In the same way as in the proof of (4.46), we deduce that for $2^k \gg 1$

$$\left| \int_\delta^{T_2} \int_{T_1}^{-\delta} \left\langle P_{\leq k} \left(e^{it\sqrt{-\Delta}} \nabla (|x|^{-3} * |u|^2) u(t) \right), P_{\leq k} \left(e^{i\tau\sqrt{-\Delta}} \nabla (|x|^{-3} * |u|^2) u(\tau) \right) \right\rangle_{L_x^2} dt d\tau \right| \lesssim 1. \tag{4.56}$$

Using the same argument as deriving (4.35), we obtain (4.52) by making use of (4.54), (4.55) and (4.56) to replace (4.42), (4.43) and (4.46). Therefore, we conclude Proposition 1.1.

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